3.7 Polynomial Function Models and the First Derivative

SETTING THE STAGE

This chapter has discussed using the derivative to determine the instantaneous rate of change of a quantity. The derivative has wide application in fields such as economics, science, engineering, and the social sciences. Here is an example:

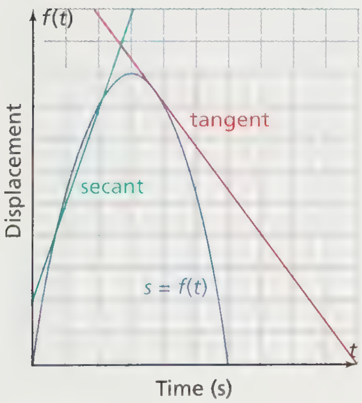
Suppose a rock is thrown into the air from a bridge 15 m above the water. As it rises and then falls, its height above the water is a function of the time since it was thrown. The height of the rock, in metres, above the water at t seconds can be modelled by the function $h(t) = -4.9t^2 + 12t + 15$.

What is the velocity of the rock when it enters the water?

In this section we will examine applications of the first derivative of a polynomial model.

EXAMINING THE CONCEPT

Polynomial Models Involving Velocity and Speed



Displacement versus time

In many situations, an object's position, s, can be described by a function of time, $s = f(t)$. Average velocity is defined as the rate of change of displacement over an interval of time. Instantaneous velocity is the rate of change of displacement at a specific point in time.

On a displacement-time graph, the slope of a secant represents average velocity, while the slope of a tangent represents instantaneous velocity.

$$\text{average velocity} = \frac{\Delta s}{\Delta t}$$

$$\begin{aligned}\text{instantaneous velocity} &= \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} \\ &= \lim_{h \to 0} \frac{f(t + h) - f(t)}{h} \\ &= f'(t)\end{aligned}$$

As a result, the derivative of the position function, $s' = f'(t)$, represents the instantaneous velocity of the object at time t. So

$$\begin{aligned}v(t) &= \frac{ds}{dt} \\ &= s' \\ &= f'(t)\end{aligned}$$

If displacement is measured in metres and time in seconds, the units of velocity or speed are metres per second (m/s).

The Difference between Velocity and Speed

The **velocity** of an object measures how fast it is moving and the direction of movement.

Speed is the magnitude or absolute value of the velocity, without regard to direction.

Example 1 Analyzing the Motion of a Falling Object: Vertical Motion

A rock is tossed from a bridge 15 m above the water. The height of the rock, h, in metres above the water at t seconds can be modelled by the function $h(t) = -4.9t^2 + 12t + 15$.

(a) Determine the instantaneous velocity at 1 s and at 2 s.

(b) What is the velocity of the rock when it enters the water?

(c) Determine the initial velocity of the rock.

(d) When is the rock at its maximum height? What is the maximum height?

Solution

(a)
$$\begin{aligned}\text{instantaneous velocity} &= v(t)\\ &= h'(t) \text{ or } \frac{dh}{dt}\\ &= \frac{d}{dt}(-4.9t^2 + 12t + 15)\\ &= -9.8t + 12\end{aligned}$$

$$\begin{aligned}v(1) &= h'(1)\\ &= -9.8(1) + 12\\ &= 2.2\end{aligned}$$

At 1 s, the rock is moving up at 2.2 m/s. (positive velocity = upward movement)

$$\begin{aligned}v(2) &= h'(2)\\ &= -9.8(2) + 12\\ &= -7.6\end{aligned}$$

At 2 s, the rock is moving down at 7.6 m/s. (negative velocity = downward movement)

(b) The rock hits the water when $h = 0$. Solve for t when $h = 0$.

$$\begin{aligned}h(t) &= -4.9t^2 + 12t + 15\\ 0 &= -4.9t^2 + 12t + 15\end{aligned}$$

Solve using the quadratic formula.

$$\begin{aligned}t &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\\ &= \frac{-12 \pm \sqrt{12^2 - 4(-4.9)(15)}}{2(-4.9)}\\ &= \frac{-12 \pm \sqrt{144 + 294}}{-9.8}\\ &= \frac{-12 \pm \sqrt{438}}{-9.8}\end{aligned}$$

Since t starts at 0 when the rock is thrown, the negative solution is inadmissible.

$$t_1 = \frac{-12 + \sqrt{438}}{-9.8} \doteq \frac{-12 + 20.93}{-9.8} = -0.91$$

$$t_2 = \frac{-12 - \sqrt{438}}{-9.8} \doteq \frac{-12 - 20.93}{-9.8} = 3.36$$

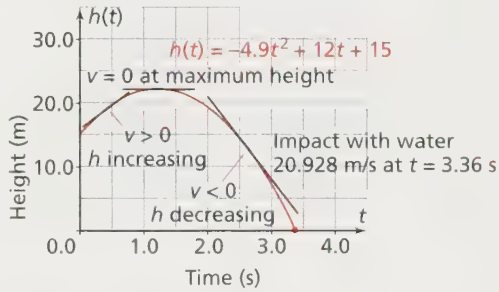
At the time of impact,
$v(t) = v(3.36) = h'(3.36) = -9.8(3.36) + 12 = -20.93$
When it hits the water, the rock is falling at 20.93 m/s.

(c) The initial velocity occurs at $t = 0$:
$v(t) = v(0) = h'(0) = -9.8(0) + 12 = 12$
The rock was thrown upward with an initial velocity of 12 m/s.

(d) A table and graph show the changing velocity.

To determine the maximum height, find the time when the instantaneous velocity is 0.

Time, t	Height, $h(t)$	Velocity, $h'(t)$
0.0	15.0	12.0
0.5	19.8	7.1
1.0	22.1	2.2
1.5	22.0	-2.7
2.0	19.4	-7.6
2.5	14.4	-12.5
3.0	6.9	-17.4



$h(t) = -4.9t^2 + 12t + 15$

Initially, the velocity is positive and the rock is moving up. The velocity starts to decrease. When the velocity is 0, the rock is motionless at its highest point. The velocity becomes a larger and larger negative value until the rock hits the water.

Solve for t when $v(t) = 0$. $v(t) = h'(t)$ and $h'(t) = -9.8 + 12$.

$$-9.8t + 12 = 0$$
$$t = \frac{-12}{-9.8}$$
$$t \doteq 1.2$$

The maximum height occurs at 1.2 s. Solve for h at $t = 1.2$.

$$\begin{aligned} h(t) &= -4.9t^2 + 12t + 15 \\ &= -4.9(1.2)^2 + 12(1.2) + 15 \\ &= 22.3 \end{aligned}$$

The rock reaches a maximum height of 22.3 m.

Example 2 Analyzing the Motion of a Moving Object: Horizontal Motion

The position of an object moving along a straight line can be modelled by the function $s(t) = 3t^3 - 40.5t^2 + 162t$, where s is the position in metres at t seconds and $t \geq 0$.

(a) Determine the initial position of the object.

(b) Determine the velocity at 2 s and 5 s.

(c) When is the object stationary?

(d) When is the object advancing? retreating?

(e) Determine the total distance travelled during the first eight seconds of motion.

Solution

(a) The initial position occurs when $t = 0$. Since $s(0) = 0$, the object starts at the origin.

(b) For $v(t)$, the velocity function,

$$\begin{aligned} v(t) &= s'(t) \\ &= \frac{d}{dt}(3t^3 - 40.5t^2 + 162t) \\ &= 9t^2 - 81t + 162 \end{aligned}$$

$$\begin{aligned} v(2) &= 9(2)^2 - 81(2) + 162 \\ &= 36 \\ v(5) &= 9(5)^2 - 81(5) + 162 \\ &= -18 \end{aligned}$$

At 2 s, the object is moving at 36 m/s. In this type of situation, assume that positive velocity means movement to the right. At 5 s, the object is moving at -18 m/s, that is, at 18 m/s to the left.

(c) The object is stationary when the velocity $v(t) = 0$. Substitute and solve for t.

$$\begin{aligned} 9t^2 - 81t + 162 &= 0 \\ 9(t^2 - 9t + 18) &= 0 \\ 9(t - 6)(t - 3) &= 0 \\ t = 6 \text{ and } t &= 3 \end{aligned}$$

The object is motionless at exactly 3 s and 6 s.

(d) For the object to be advancing, $v(t) > 0$. Substitute and then solve the inequality.

$$\begin{aligned} 9t^2 - 81t + 162 &> 0 \\ (t - 6)(t - 3) &> 0 \end{aligned}$$

The same factorization as in (c).

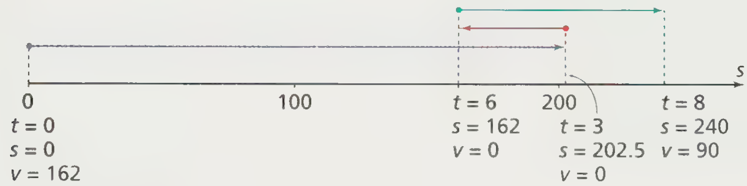
	Interval		
	$0 \leq t < 3$	$3 < t < 6$	$t > 6$
$t - 6$	–	–	+
$t - 3$	–	+	+
$v(t)$	(–)(–) = +	(–)(+) = –	(+)(+) = +

Since $t \geq 0$, there are three intervals to consider. The object is advancing when $0 \leq t < 3$ and when $t > 6$. The object is retreating when $3 < t < 6$.

(e) The total distance travelled includes both advancing and retreating motion. The total distance is the sum of the absolute distances travelled. The object changes direction at 3 s and at 6 s, and starts at the origin, $s(0) = 0$.

Time, t (s)	Position, s (m) $s(t) = 3t^3 - 40.5t^2 + 162t$	Time Interval	Distance Travelled
3	$s(3) = 3(3)^3 - 40.5(3)^2 + 162(3)$ $= 202.5$	$0 < t < 3$	$\lvert s(3) - s(0) \rvert = \lvert 202.5 - 0 \rvert = 202.5$
6	$s(6) = 162$	$3 < t < 6$	$\lvert s(6) - s(3) \rvert = \lvert 162 - 202.5 \rvert = 40.5$
8	$s(8) = 240$	$6 < t < 8$	$\lvert s(8) - s(6) \rvert = \lvert 240 - 162 \rvert = 78$

The total distance travelled between 0 s and 8 s is
202.5 m + 40.5 m + 78 m = 321 m.



Horizontal motion

EXAMINING THE CONCEPT

Other Rates of Change

Use the derivative to solve problems that involve instantaneous rates of change in a variety of applications. Often in these problems, the independent variable is time t. The variable Q in $Q = f(t)$ is a quantity that varies with time. For example, Q might be

- the size of a population
- the number of dollars in a bank account
- the volume of a balloon being inflated or deflated
- the amount of liquid in a tank that is being filled or drained
- the total distance travelled over an interval of time

Average Rate of Change

The **average rate of change** of Q is the ratio of ΔQ, which is the change in Q, to Δt, which is the change in t. The average rate of change equals the slope of the secant to the curve over the interval Δt.

$$\frac{\Delta Q}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t} \text{ or } \frac{\Delta Q}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

Instantaneous Rate of Change

The **instantaneous rate of change** of $Q = f(t)$ at time t is equal to the slope of the tangent line to the curve $Q = f(t)$ at point $(t, f(t))$.

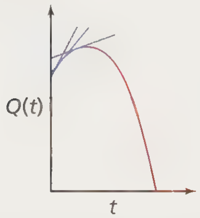
$$\frac{dQ}{dt} = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

$$\text{or } \frac{dQ}{dt} = f'(t)$$

The derivative of $Q = f(t)$ represents the instantaneous rate of change at point $(t, f(t))$.

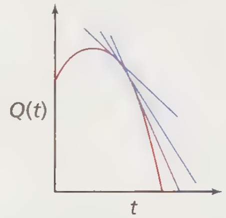
A positive slope corresponds to a rising tangent line, and a negative slope corresponds to a falling tangent line. Therefore,

Q is increasing at time t if $\frac{dQ}{dt} = f'(t) > 0$



Tangents with positive slope; *Q* increasing

Q is decreasing at time t if $\frac{dQ}{dt} = f'(t) < 0$



Tangents with negative slope; *Q* decreasing

Example 3 Analyzing a Polynomial Population Model

The world population is growing rapidly, as shown in the data from the World Population Division of the United Nations.

(a) Create a scatter plot and determine a cubic polynomial that models the data.

(b) Determine the average rate of change in the world population between 1960 and 1990.

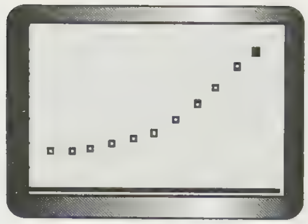
(c) Determine the rate at which the world population is changing in 1995.

(d) Use graphing technology to analyze the yearly growth rate of the world population since 1990.

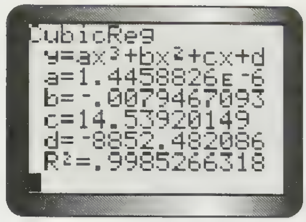
Year	1900	1910	1920	1930	1940	1950	1960	1970	1980	1990	1998	1999
World Population (billions)	1.65	1.75	1.86	2.07	2.30	2.52	3.02	3.70	4.44	5.27	5.90	6.00

Solution

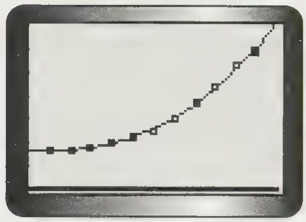
(a) Using a TI-83 Plus calculator, enter the year data into **L1** and the world population data into **L2**. Turn the **STAT PLOTS** on to create the scatter plot.



Perform a cubic regression on the data.



Graph the **CubicReg** function.



A cubic polynomial model of this data, for $1900 \leq t \leq 1999$, is

$$f(t) = 0.000\,001\,445\,882\,6t^3 - 0.007\,946\,709\,3t^2 + 14.539\,201\,49t - 8852.482\,086$$

(b) For the specified period, $t_1 = 1960$ and $t_2 = 1990$.

$$\begin{aligned}\text{average rate of change} &= \frac{f(t_2) - f(t_1)}{t_2 - t_1}\\ &= \frac{f(1990) - f(1960)}{1990 - 1960}\\ &\doteq \frac{5.186\,358\,8 - 3.099\,476\,0}{30}\\ &\doteq \frac{2.086\,875\,9}{30}\\ &\doteq 0.069\,562\,53\end{aligned}$$

Between 1960 and 1990, the world population increased by an average of about 70 million people per year.

(c) The instantaneous rate of change is

$$\begin{aligned}f'(t) &= 0.000\,004\,337\,647\,8t^2 - 0.015\,893\,418\,6t + 14.539\,201\,49\\ f'(1995) &= 0.000\,004\,337\,647\,8(1995)^2 - 0.015\,893\,418\,6(1995) + 14.539\,201\,49\\ &\doteq 0.095\,778\end{aligned}$$

In 1995, the world population was increasing at the rate of about 96 million per year.

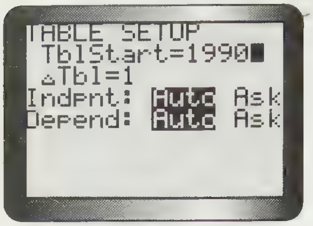
(d) Enter the cubic regression equation, $f(t)$, into **Y1** of the equation editor and its derivative, $f'(t)$, into **Y2**.

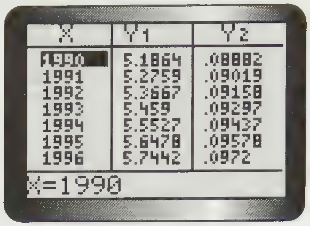
Technology Help

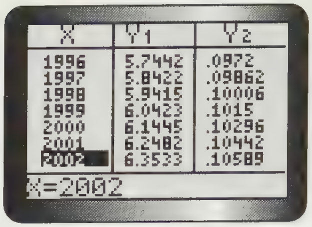
The derivative can be entered using the **nDeriv(** operation. For detailed instructions, see page 595 of the Technology Appendix.

Press [2nd] [TBLSET] and set up a table as shown.

Press [2nd] [TABLE] to display the table.







The world population is displayed in **Y1** and the yearly growth rate in **Y2**. From 1990 to 2002, the rate at which the population is growing increases every year.

CHECK, CONSOLIDATE, COMMUNICATE

1. What does the derivative of the function represent if the position of an object can be modelled by a function of time?
2. When you throw an object into the air, how can you use a derivative to determine when the object reaches its maximum height?
3. What does a negative rate of change indicate? What does a positive rate of change indicate?

KEY IDEAS

- The **average rate of change** of $Q = f(t)$ is the ratio of the change in Q to the change in t. The average rate of change equals the slope of the secant to the curve over the interval Δt.

$$\frac{\Delta Q}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t} \quad \text{or} \quad \frac{\Delta Q}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

- The **instantaneous rate of change** of $Q = f(t)$ at time t is equal to the derivative of Q with respect to t. This is equivalent to the slope of the tangent line to the curve $Q = f(t)$ at point $(t, f(t))$.

$$\frac{dQ}{dt} = f'(t)$$

- A positive slope corresponds to a rising tangent line, and a negative slope corresponds to a falling tangent line. Therefore,
  - Q is increasing at time t if $\frac{dQ}{dt}$ or $f'(t) > 0$
  - Q is decreasing at time t if $\frac{dQ}{dt}$ or $f'(t) < 0$

- **Horizontal Motion** If the displacement, $s(t)$, of an object moving horizontally along a line is a function of time, s, the derivative represents the instantaneous velocity $v(t)$.

$$v(t) = \frac{ds}{dt} = s'(t) = f'(t)$$

If $v(t) > 0$, it represents movement to the right of its initial position. If $v(t) < 0$, it represents movement to the left. If $v(t) = 0$, the object is stationary and may be in the process of changing direction.
- **Vertical Motion** If the height of a falling object is a function of time, $h = f(t)$, the derivative represents the instantaneous velocity $v(t)$. If $v(t) > 0$, the object is travelling up. If $v(t) < 0$, the object is falling down. If $v(t) = 0$, the object is at its maximum height.

3.7 Exercises

A

1. Stand in the centre of an open space, noting your starting position. Take three steps forward, followed by one step back.
   (a) How many steps are you from your starting point? The number of steps is your displacement.
   (b) How many steps have you taken? The number of steps is the distance you travelled.

2. When any object is in motion, three related characteristics of the motion can be measured. Copy and complete the table. Indicate the appropriate units.

Displacement, $f(t)$	Velocity, $f'(t)$	Time, t
	kilometres per hour	
metres		seconds
	metres per minute	
centimetres		seconds

3. For each function, find an expression for the velocity and the value of the displacement and velocity at $t = 5$. Displacement is in metres and time is in seconds. Include the appropriate units in your responses.
   (a) $s(t) = t^2 - 4t + 5$
   (b) $s(t) = 3t + 7$
   (c) $s(t) = 12$
   (d) $s(t) = t^3 - 2t^2 + 4t - 1$

B

4. An object is projected directly up so that its height in metres at time t seconds can be modelled by $h(t) = -0.5t^2 + 9t + 9.1$.
   (a) From what height was the object initially projected?
   (b) What was the initial velocity?

(c) Find the velocity when $t = 2, 5, 9$, and 11 s.

(d) Find the height when $t = 8, 9$, and 10 s.

(e) When does the object return to its initial height?

(f) Sketch the path of the object and a height-versus-time graph of the motion.

5. When a flare is launched from an oceanside hilltop, its height above the water, in metres, at t seconds is described by $h(t) = -4.9t^2 + 16t + 200$.

   (a) What is the initial velocity of the flare?

   (b) What is the height at launch?

   (c) How fast is the flare descending at 6 s after launch?

   (d) When does the flare hit the water?

   (e) What is its velocity when it hits the water?

6. **Knowledge and Understanding**: A model rocket is launched with an initial velocity of 55 m/s. Its height as a function of time can be modelled by $h(t) = 55t - 4.9t^2$. Determine the maximum height reached by the rocket.

7. The population of a city has been tracked since 1980. The population growth, P, is a function of the number of years after 1980, x.

$$P(x) = 2(x - 20)^3 + 20\,000, 0 \leq x \leq 40$$

   (a) Use your knowledge of transformations to sketch the function.

   (b) What is the projected population in 2010?

   (c) At what rate is the population growing between 1995 and 2010?

   (d) At what rate is the population expected to grow in 2012?

   (e) Is the growth rate the same at any other time?

8. An environmental report estimates that acid rain is changing the conditions in a lake so that the fish population in hundreds, f, as a function of t years since 1990 can be described by $f(t) = 1200 + 150t - 15t^2$.

   (a) How was the fish population changing in 2000? in 1992? in 1997?

   (b) What was the average rate of change in the first five years covered by the report?

   (c) If no environmental improvements occur, when will the fish population become 0?

9. A particle moves along a line. The particle's position, s, in metres at t seconds is modelled by $s(t) = 2t^3 - 15t^2 + 36t + 40$, where $t \geq 0$.

   (a) Determine the initial position of the particle.

   (b) What is the velocity at 1 s? at 5 s?

   (c) When is the particle stationary?

   (d) When is it advancing? retreating?

   (e) Determine the total distance travelled during the first five seconds.

10. A particle moves along a line. The particle's position, s, in centimetres at t seconds is modelled by $s(t) = t^3 - 9t^2 + 24t + 20$, where $t \geq 0$.

(a) What is the total distance travelled by the particle in the first 8 s?

(b) Draw a diagram that illustrates the particle's motion in the first 8 s.

11. The position of a remote-control car as it moves back and forth along a straight path is given in the table.

Time (s)	0	1.0	1.3	2.0	2.3	3.0	3.5	4	4.5	5	5.1	5.3	5.6	6.1
Displacement (m)	78.0	16.0	6.2	–4.0	–4.5	0.0	5.8	10.0	11.6	8.0	5.8	2.4	–7.1	–29.0

(a) Develop an algebraic model for the motion.

(b) Find the initial position of the car compared with the fixed point.

(c) When is the car moving away from the fixed point?

(d) What is the car's velocity at $t = 3$?

(e) What is the average velocity from $t = 1$ to $t = 5$?

12. The table describes the flight of a toy glider launched from a tower on a hilltop. The values for height indicate the number of metres above or below the top of the hill.

Time (s)	0	1	2	3	4	5	6	7	8	9
Height (m)	9.0	5.5	2.5	0.0	–2.0	–3.5	–4.5	–5.0	–5.0	–4.5

Time (s)	10	11	12	13	14	15	16	17	18
Height (m)	–3.5	–2.0	0.0	2.5	5.5	9.0	13.0	17.5	22.5

(a) Find an algebraic model that gives height as a function of time.

(b) Find the initial velocity.

(c) Find the average velocity between 4 s and 7 s.

(d) Find the velocity at 15 s.

(e) Describe the position and direction of the glider at 6 s, 12 s, and 14 s.

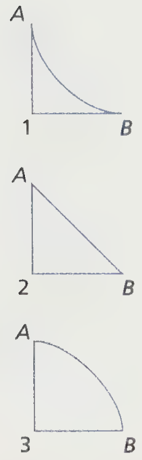
13. **Thinking, Inquiry, Problem Solving:** The table shows the percentage of Canadians who are between 15 years old and 19 years old and who smoke.

Year	1981	1983	1985	1986	1989	1991	1994	1995	1996
Males (%)	43.4	39.6	26.7	25.2	22.6	22.6	27.3	28.5	29.1
Females (%)	41.7	40.5	27.7	27.0	23.5	25.6	28.9	29.5	31.0

Source: Health Canada

(a) Determine when the rate of change for male smokers is exactly the same as the rate of change for female smokers.

(b) Is the percentage of smokers increasing or decreasing at the present time? Explain.



Ramp profiles

14. **Communication:** Each diagram on the left shows the side profile of a different ramp. Imagine a ball rolling down each ramp and stopping at B. For each ramp, A and B are in the same relative positions.

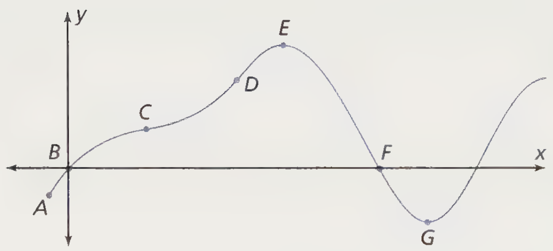
(a) On which ramp does the ball travel the longest distance? the shortest? Explain your reasoning.

(b) On which ramp is the ball travelling the fastest one-quarter of the way down the ramp? the slowest? Explain.

(c) On which ramp will the speed of the ball at the bottom of the ramp be the greatest? the least? Explain.

(d) For each ramp, sketch a graph of the speed of the ball as a function of time.

15. For each section of the graph, complete the corresponding line in the table. Indicate whether the value of the function in that interval is positive or negative (+ or −). Also indicate whether the values are increasing or decreasing (↗ or ↘) in each interval. Indicate the same information for the derivative.



	Value of Function		Value of Derivative	
Section	+ or −	↗ or ↘	+ or −	↗ or ↘
A to B				
B to C				
C to D				
D to E				
E to F				
F to G				

16. **Application:** A bucket that initially held 5 L of water has a leak. After t seconds, the amount of water, Q, in litres remaining in the bucket is represented by $Q(t) = 5\left(1 - \frac{t}{25}\right)^2$.

(a) How fast, to the nearest hundredth, is the water leaking from the bucket at 3 s?

(b) How long does it take for all of the water to leak out of the bucket?

(c) At what rate is the water leaking when the last drop leaks out?

17. An environmental study of a suburban community suggests that, t years from now, the level of carbon monoxide in the air, measured in parts per million, can be modelled by the function $q(t) = 0.05t^2 + 0.2t + 2.7$.

(a) At what rate will the carbon monoxide level be changing with respect to time two years from now?

(b) Does this model predict an increase or a decrease in the carbon monoxide level over the long term? Explain.

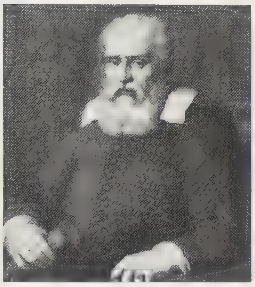
18. The position, s, in metres of a moving particle is given by $s(t) = 2t^3 - 2t^2 - 5t$, where $t \geq 0$. When does the particle reach a velocity of 8 m/s? Anwer to three decimal places.

19. **Check Your Understanding**

(a) Draw a graph that shows an object's displacement increasing over time. What can you say about the object's velocity in this situation?

(b) Draw a graph that shows the object's displacement decreasing over time. What can you say about the object's velocity in this situation?

C 20. Consider $f(x) = (x - 3)(x + 4)(2x + 15)$. How quickly is the value of the function changing when $x = 1$?



Galileo Galilei (1564–1642)

Before the time of Galileo, the physical world was described in terms of qualities, such as potentiality. Galileo advanced science by using measurable quantities like time, distance, and mass, to describe things.

21. (a) For $s(t) = t^2 + 8t$, where s is in metres and t is in seconds, find the average velocity between $t = 2$ and $t = 3$.

(b) Find the arithmetic average of the velocities at $t = 2$ and $t = 3$.

(c) Find the instantaneous velocity at $t = 2.5$.

(d) Repeat (a) to (c) for the displacement function $s(t) = t^3 + 9t$.

(e) Explain why the three values are equal for the first function but different for the second function.

22. Two objects start from the same location at the same time and move along the same straight path. After t seconds, their displacements from the starting point are given by $f(t) = 2t^2 - 3t$ and $g(t) = 3t - t^2$.

(a) Determine when the objects have equal velocities.

(b) What are the velocities when the object's positions are the same?

23. A car is travelling at 80 km/h when the driver applies the brakes to avoid a moose on the road. After t seconds, the car is $s(t) = 80t - 3t^2$ metres from the point where the brakes were first applied. How far does the car travel before it stops?

ADDITIONAL ACHIEVEMENT CHART QUESTIONS

Knowledge and Understanding: The distance a test car has travelled at t seconds is represented by $d(t) = 12t + t^2$, where d is the distance in metres.

(a) Find the velocity at time t in metres per second.

(b) How fast is the test car travelling at 10 s?

(c) When does the test car reach a velocity of 60 m/s?

Application: A ball is thrown up so that its height is represented by $H(t) = 20t - 5t^2$, where H represents the height in metres after t seconds. After starting to descend, the ball strikes a post at a point 15 m high. Find the rate of descent of the ball at the moment when the ball strikes the post. For how long is the ball rising? falling? Sketch a graph to see if your findings are reasonable.

Thinking, Inquiry, Problem Solving: Two objects, A and B, begin to move from the same starting point, at the same time, and travel along the same straight line. After t seconds, their displacements, in metres, from the starting point are modelled by $s(t) = 2t^2 - 5t + 2$ and $g(t) = -3t^2 + t + 3$, respectively, where $t \geq 0$.

(a) When is A moving faster than B? slower than B?

(b) A comes to a sudden stop. How fast is B travelling at that time?

(c) When does B change directions? How can you tell?

Communication: One of your classmates takes calculus, but not physics. When a problem involving vertical motion is discussed, he has trouble using the function model to find **(a)** the height the object is above a surface, **(b)** the initial velocity at which the object is thrown, and **(c)** at what time the object reaches the ground surface. Create a vertical motion problem. Then use visual models, diagrams, and graphs to clearly explain this problem. Also show how to use the function model to solve this problem.

The Chapter Problem

Average Salaries in Professional Sports

Apply what you learned to answer these questions about The Chapter Problem on page 168.

CP10. Using the derivatives (question CP7), determine the instantaneous rate of change for each season for each data set. (Hint: Consider using a spreadsheet or the lists in a graphing calculator.) When are the salaries increasing the fastest? the slowest?

CP11. Using the cubic regression equations (question CP7), predict the average salary for each league in 2005. How well do you think the model will reflect the actual average salary at that time? Explain.

CP12. Research the average salaries of other sports. Ensure that your research covers the same period. Compare the data, and analyze the rates of change as you did for the original data. Discuss high or low rates of change that you notice in the new data. Write a report to summarize your findings and discuss trends in the new data.

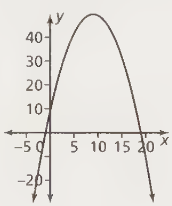
14. (a) i. $\left(\frac{1}{5}, \frac{1}{5}\right)$ ii. $\left(-\frac{1}{4}, -\frac{13}{4}\right)$ iii. $\left(\frac{1}{3}, \frac{103}{27}\right)$ and $(5, -47)$
(b) At these points, the slope of the tangent to the curve is zero (the rate of change of the value of the function with respect to the domain is zero).
15. $a = -4$, $b = 32$, $c = 0$
16. (a) 100 (b) 1200 (c) 370 bacteria per hour
17. (a) 2.857 87 cups/day (b) 0.210 845 cups/day per day
18. (a) −188
(b) slope of tangent to $f(x)$ at $x = 3$; rate of change in value of $f(x)$ with respect to x at $x = 3$
20. $(-2, -21)$ and $(-2, 6)$. Points may vary.

3.7 Exercises, page 241

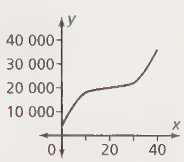
1. (a) 2 forward (b) 4
2.

Displacement $[f(t)]$	Velocity $[f'(t)]$	Time $[t]$
kilometres	kilometres per hour	hour
metres	metres per second	second
metres	metres per minute	minute
centimetres	centimetres per second	second

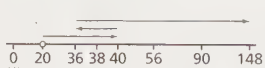
3. (a) $v(t) = 2t - 4$; 10 m; 6 m/s
(b) $v(t) = 3$; 22 m; 3 m/s
(c) $v(t) = 0$; 12 m; 0 m/s
(d) $v(t) = 3t^2 - 4t + 4$; 94 m; 59 m/s
4. (a) 9.1 m (b) 9 m/s
(c) 7 m/s; 4 m/s; 0 m/s; −2 m/s (d) 49.1 m; 49.6 m; 49.1 m
(e) after 18 s (f)



5. (a) 16 m/s (b) 200 m (c) −42.8 m/s
(d) 8.2 s (e) −64.6 m/s
6. 154.3 m
7. (a)

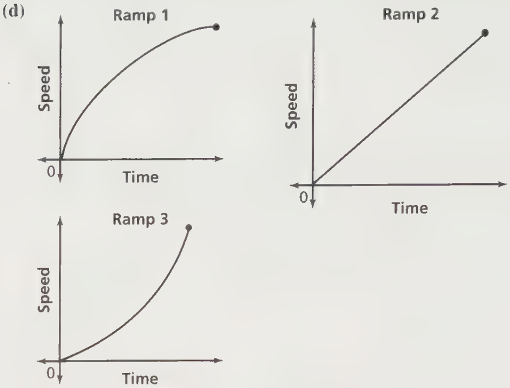


(b) 22 000 people
(c) 150 people/year (d) 864 people/year (e) yes, in 1988
8. (a) −150 fish/year; 90 fish/year; −60 fish/year
(b) 75 fish/year (c) between 2005 and 2006
9. (a) 40 m (b) 12 m/s; 36 m/s (c) 2 s or 3 s
(d) advancing: $0 \le t < 2$ and $t > 3$; retreating: $2 < t < 3$
(e) 57 m
10. (a) 136 cm (b)



11. (a) $-3t^3 + 29.9t^2 - 88.7t + 77.9$
(b) 78 (c) $2.235 < t < 4.409$
(d) 9.7 m/s (e) −2 m/s
12. (a) $h(t) = 0.25t^2 - 3.75t + 9$ (b) −3.75 m/s
(c) −1 m/s (d) 3.75 m/s
(e) 4.5 m below moving down; at hill's height moving up; 5.5 m above moving up
13. (a) between 1990 and 1991 (b) increasing

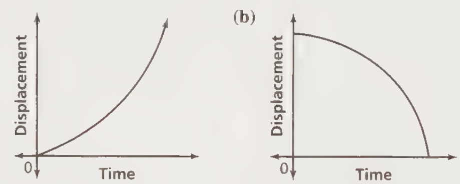
14. (a) farthest: 1 and 3 (curved path); shortest: 2 (Pythagoras)
(b) fastest: 1 (large slope of tangent); slowest: 2 (small slope of tangent)
(c) fastest: 3 (large slope of tangent); slowest: 1 (small slope of tangent)
(d)



15.

Section	Value of Function		Value of First Derivative	
	+ or	↗ or ↘	+ or −	↗ or ↘
A to B	−	↗	+	0
B to C	+	↗	+	↘
C to D	+	↗	+	↗
D to E	+	↗	+	↘
E to F	+	↘	−	↘
F to G	−	↘	−	↗

16. (a) −0.35 L/s (b) 25 s (c) 0 L/s
17. (a) 0.4 parts per million/year
(b) increase; first derivative always pos.
18. 1.843 s
19. Answers will vary. Examples:
(a) (b)



pos. velocity neg. velocity

20. 31
21. (a) 13 m/s (b) 13 m/s
(c) 13 m/s (d) 28 m/s; 28.5 m/s; 27.75 m/s
(e) the first velocity function is linear and the second is not
22. (a) $t = 1$ s
(b) at $t = 0$ s: −3 m/s and 3 m/s; at $t = 2$ s: 5 m/s and −1 m/s
23. $\frac{1600}{3}$ m

3.8 Exercises, page 251

1. (a) $f'(x) = 80x^3$; $f''(x) = 240x^2$
(b) $g'(x) = -72x^5 - 6x$; $g''(x) = -360x^4 - 6$
(c) $y' = 4x^3 + 6x^2 - 10x$; $y'' = 12x^2 + 12x - 10$
(d) $h'(x) = 12x^3 - 12x^2 - 6x$; $h''(x) = 36x^2 - 24x - 6$
(e) $y' = 8x - 3$; $y'' = 8$
(f) $y' = 5$; $y'' = 0$
(g) $f'(x) = 0$; $f''(x) = 0$
(h) $y' = -12x^{-4} + 6x^2$; $y'' = 48x^{-5} + 12x$
2. (a) 105 (b) 3 (c) −6 (d) −78
(e) 3 (f) 1448 (g) $-\frac{202}{27}$ (h) $-\frac{185}{16}$