4.4 Finding Some Shortcuts—The Product Rule

SETTING THE STAGE

In section 3.6, you saw that the derivative of the sum or difference of two functions was the sum or difference of the derivatives of the two functions. Is it also true that the derivative of the product of two functions is the product of their derivatives?

Consider the function $f(x) = 10x^9$. By the power rule, $f'(x) = 90x^8$. What happens if you factor $f(x)$ and multiply the derivatives of the factors? Let $h(x) = 5x^4$ and $g(x) = 2x^5$, so that $f(x) = h(x)g(x)$.

Differentiating h and g and multiplying gives

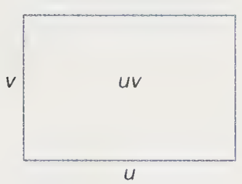
$$\begin{aligned} h'(x)g'(x) &= (20x^3)(10x^4) \\ &= 200x^7 \end{aligned}$$

Clearly, the results are different. The derivative of a product of two functions is *not* the product of the derivatives. If $f(x) = h(x)g(x)$, then $f'(x) \neq h'(x)g'(x)$.

In this section, you will develop a rule to differentiate the product of two functions.

EXAMINING THE CONCEPT

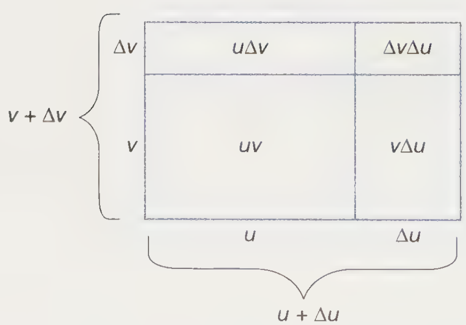
Deriving the Product Rule



Let u and v represent the functions $f(x)$ and $g(x)$, respectively. Assume that both these functions are positive and differentiable at x. Then write the product $f(x)g(x)$ as uv.

Think of the product uv as the area of a rectangle with dimensions u by v.

Let x change by a small amount, Δx. Then $f(x)$ and $g(x)$ change to $f(x + \Delta x)$ and $g(x + \Delta x)$. In terms of u and v,



$$\begin{aligned} \Delta u &= f(x + \Delta x) - f(x) \text{ and} \\ \Delta v &= g(x + \Delta x) - g(x) \end{aligned}$$

Now think of the product $(u + \Delta u)(v + \Delta v)$ as the area of the largest rectangle on the right, assuming that Δu and Δv are positive.

The change in x causes a change in the area of the original rectangle.

$$\begin{aligned} \Delta(uv) &= (u + \Delta u)(v + \Delta v) - uv \\ &= uv + u\Delta v + v\Delta u + \Delta u\Delta v - uv \\ &= u\Delta v + v\Delta u + \Delta u\Delta v \end{aligned}$$

Divide $\Delta(uv)$ by Δx to find the rate of change in area with respect to x.

$$\begin{aligned}\frac{\Delta(uv)}{\Delta x} &= \frac{u\Delta v + v\Delta u + \Delta u\Delta v}{\Delta x} \\ &= u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}\end{aligned}$$

Now let $\Delta x \to 0$ to determine an expression for the derivative of the product uv. Apply the limit laws to simplify the expression.

$$\begin{aligned}\frac{d}{dx}(uv) &= \lim_{x \to 0} \frac{\Delta(uv)}{\Delta x} \\ &= \lim_{\Delta x \to 0}\left(u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}\right) \\ &= u\lim_{\Delta x \to 0}\frac{\Delta v}{\Delta x} + v\lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x} + \left(\lim_{\Delta x \to 0}\Delta u\right)\left(\lim_{\Delta x \to 0}\frac{\Delta v}{\Delta x}\right)\end{aligned}$$

Apply the definition of the derivative. Since u is differentiable at x and therefore continuous, $\Delta u \to 0$ as $\Delta x \to 0$.

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} + 0\left(\frac{dv}{dx}\right)$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}(uv) = \frac{du}{dx}v + \frac{dv}{dx}u$$

The result is known as the **product rule** for differentiable functions. Although this derivation assumed positive quantities, the result is true whether u, v, Δu, and Δv are positive or negative. Replace u with $f(x)$ and v with $g(x)$ to express the product rule in another way.

The derivative of the product of two functions is the sum of the derivative of the first function times the second function, plus the derivative of the second function times the first function.

The Product Rule

Given $h(x) = f(x)g(x)$ and $f(x)$ and $g(x)$ are both differentiable, then $h'(x) = f'(x)g(x) + g'(x)f(x)$. In Leibniz notation,

$$\frac{d[h(x)]}{dx} = \frac{d[f(x)]}{dx}\cdot g(x) + \frac{d[g(x)]}{dx}\cdot f(x)$$

Example 1 Using the Product Rule to Find the Derivative

Determine the derivative using the product rule.

(a) $y = 3x^4(5x^3 + 5x - 7)$ **(b)** $y = (x^4 - 4x^3 - 2x^2 + 5x + 2)^2$

Solution

(a) Use the product rule. Let $f(x) = 3x^4$ and $g(x) = 5x^3 + 5x - 7$.

$$\frac{dy}{dx} = \frac{d}{dx}[3x^4(5x^3 + 5x - 7)]$$ Apply the product rule.

$$= \left[\frac{d}{dx}(3x^4)\right](5x^3 + 5x - 7) + \left[\frac{d}{dx}(5x^3 + 5x - 7)\right]3x^4$$ Differentiate polynomials.

$$= 12x^3(5x^3 + 5x - 7) + (15x^2 + 5)3x^4$$ Expand and simplify.

$$= 60x^6 + 60x^4 - 84x^3 + 45x^6 + 15x^4$$
$$= 105x^6 + 75x^4 - 84x^3$$

(b) $y = (x^4 - 4x^3 - 2x^2 + 5x + 2)^2$

$$= (x^4 - 4x^3 - 2x^2 + 5x + 2)(x^4 - 4x^3 - 2x^2 + 5x + 2)$$

$$\frac{dy}{dx} = \left[\frac{d}{dx}(x^4 - 4x^3 - 2x^2 + 5x + 2)\right](x^4 - 4x^3 - 2x^2 + 5x + 2)$$
$$+ \left[\frac{d}{dx}(x^4 - 4x^3 - 2x^2 + 5x + 2)\right](x^4 - 4x^3 - 2x^2 + 5x + 2)$$
$$= (4x^3 - 12x^2 - 4x + 5)(x^4 - 4x^3 - 2x^2 + 5x + 2)$$
$$+ (4x^3 - 12x^2 - 4x + 5)(x^4 - 4x^3 - 2x^2 + 5x + 2)$$
$$= 2(x^4 - 4x^3 - 2x^2 + 5x + 2)(4x^3 - 12x^2 - 4x + 5)$$

In both parts of this example, you can also find the derivative without using the product rule. First expand the product and then simplify. Differentiate using the rules for polynomials. This method would be easier for differentiating the product in (a). You will encounter more complex functions later on, where using the product rule is the only practical way to find the derivative.

Example 2 Using the Product Rule to Find the Equation of a Tangent Line

Find the equation of the tangent line to $f(x) = (2x + 4)(3x^3 - 3x^2 + x - 2)$ at point $(1, -6)$.

Solution

Differentiate to find the slope of the tangent line. You could expand first before differentiating, but it is easier to use the product rule.

$$f'(x) = \left[\frac{d}{dx}(2x + 4)\right](3x^3 - 3x^2 + x - 2) + \left[\frac{d}{dx}(3x^3 - 3x^2 + x - 2)\right](2x + 4)$$
$$= (2)(3x^3 - 3x^2 + x - 2) + (9x^2 - 6x + 1)(2x + 4)$$

The slope of the tangent line at $x = 1$ is $f'(1)$. In this case, it is easier to substitute $x = 1$ into the derivative before simplifying to obtain the slope.

$$f'(1) = (2)[3(1)^3 - 3(1)^2 + (1) - 2] + [9(1)^2 - 6(1) + 1][2(1) + 4]$$
$$= (2)(-1) + (6)(4)$$
$$= 22$$

Determine the equation by substituting $m = 22$ and point $(1, -6)$ into $y = mx + b$.

$$-6 = 22(1) + b \qquad \text{Solve for } b.$$
$$-28 = b$$

Therefore, the equation of the tangent line is $y = 22x - 28$.

Example 3 Using the Product Rule to Graph a Function

For $f(x) = (4x - 8)(2x^2 + 2x + 4)$, determine the critical numbers, the intervals of increase or decrease, and the extrema, then graph $f(x)$.

Solution

$$f'(x) = \left[\frac{d}{dx}(4x - 8)\right](2x^2 + 2x + 4) + \left[\frac{d}{dx}(2x^2 + 2x + 4)\right](4x - 8)$$
$$= 4(2x^2 + 2x + 4) + (4x + 2)(4x - 8)$$
$$= 24x^2 - 16x$$

To find the critical numbers, solve $f'(x) = 0$.

$$24x^2 - 16x = 0$$
$$8x(3x - 2) = 0$$
$$x = 0 \text{ or } x = \frac{2}{3}$$

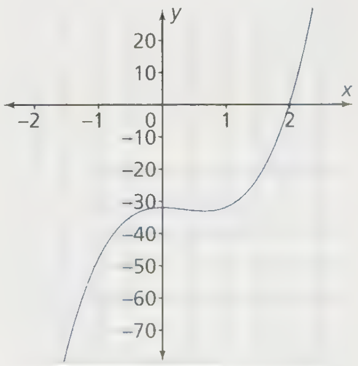
	Intervals		
	$x < 0$	$0 < x < \frac{2}{3}$	$x > \frac{2}{3}$
$8x$	–	+	+
$3x - 2$	–	–	+
$f'(x)$	$(-)(-) = +$	$(+)(-) = -$	$(+)(+) = +$
$f(x)$	increasing ↗	decreasing ↘	increasing ↗
	maximum at $x = 0$		minimum at $x = \frac{2}{3}$

Evaluate $f(x) = (4x - 8)(2x^2 + 2x + 4)$ at the critical numbers.

$$f(0) = [4(0) - 8][2(0)^2 + 2(0) + 4]$$
$$= -32$$

$$f\left(\frac{2}{3}\right) = \left[4\left(\frac{2}{3}\right) - 8\right]\left[2\left(\frac{2}{3}\right)^2 + 2\left(\frac{2}{3}\right) + 4\right]$$
$$= -33\frac{5}{27}$$

The function has a local maximum value, $f(0) = -32$ and a local minimum value, $f\left(\frac{2}{3}\right) = -33\frac{5}{27}$. The function is decreasing for $0 < x < \frac{2}{3}$ and increasing for all other values of x, where $x \in \mathbf{R}$. With this information, graph $f(x)$.



$f(x) = (4x - 8)(2x^2 + 2x + 4)$

CHECK, CONSOLIDATE, COMMUNICATE

1. State the product rule in your own words.
2. For $f(x) = (3x^2 + 5x - 4)(x - 1)$, find the derivative by using the product rule. Then find the derivative by first expanding and then differentiating the polynomial. Verify that the derivatives are the same.

KEY IDEAS

- The product rule for differentiation is summarized below.

Function Notation	Leibniz Notation
Given $h(x) = f(x)g(x)$ and f and g are both differentiable, then $h'(x) = f'(x)g(x) + g'(x)f(x)$.	Given $h(x) = f(x)g(x)$, and f and g are both differentiable, then $\frac{d}{dx}[h(x)] = \frac{d}{dx}[f(x)] \cdot g(x) + \frac{d}{dx}[g(x)] \cdot f(x)$

- In some cases, it is easier to expand and simplify the product before differentiating rather than using the product rule.

4.4 Exercises

A

1. Use the product rule to state the derivative of each function. Write your answers in unsimplified form.
   (a) $f(x) = x(x^3)$
   (b) $h(x) = 2x^2(x^2 - 2x)$
   (c) $f(x) = (3 - 2x^3)(5 + 4x)$
   (d) $y = (3x^2 + 4x - 6)(2x^2 - 9)$
   (e) $g(x) = (4x^4 - 8x^2)^2$
   (f) $y = (4x^3 - 2x^2 + 8x)(-x^2 - 5x + 3)$
2. Let $F(x) = [b(x)][c(x)]$. Express $F'(x)$ in terms of $b(x)$ and $c(x)$.
3. (a) Let $f(x) = 5x^4(3x^3 - 4x^2 + 6x - 7)$. Find $f'(x)$ using the product rule.
   (b) Expand $f(x)$ and then find its derivative.
   (c) Compare the results for (a) and (b).

4. (a) Let $y = (3x^2 - 7x)(-5x^2 - 8x + 3)$. Expand the expression and then find its derivative.
(b) Find $\frac{dy}{dx}$ using the product rule.
(c) Compare the results for (a) and (b).

B

5. Find the derivative.
(a) $f(x) = (5x^3 - 6x)(x^2 + 7)$
(b) $f(t) = (2t^2 + 4t - 5)(3t + 2)$
(c) $f(x) = (3x^2 - 6x)^2$
(d) $f(x) = x^3(x^2 - 3)(x^2 + 3)$
(e) $f(x) = 5x^{-2}(7 - x^{-3})$
(f) $f(z) = \sqrt{z}(z^4 - 2z)$
(g) $f(t) = \left(2t - t^{\frac{1}{2}}\right)\left(5t^{\frac{3}{2}} + 4t\right)$
(h) $f(z) = z^{-2}\left(z^{\frac{1}{3}} + z^{\frac{2}{3}} + 4\right)$

6. **Knowledge and Understanding:** For $y = (3x^3 - 2x^2 + x - 1)^2$, find $\frac{dy}{dx}$ without first expanding the function.

7. Given $f(x) = g(x)h(x)$, $g(x) = x$, and $h(x) = x^2$, graph $g(x)$, $h(x)$, $f(x)$, and $f'(x)$ on the same set of axes.

8. Determine the slope of the tangent line at the indicated x-value.
(a) $f(x) = (2x - x^2)(x^2 - 1)$ at $x = 1$
(b) $f(x) = (x^2 + 6x - 3)(-4x + 1)$ at $x = -2$
(c) $f(x) = (5x^3 - x)^2$ at $x = 0$
(d) $f(x) = 4x^3(x^2 - 2x + 3)$ at $x = 3$
(e) $f(x) = 2x^{-1}(6x - 2x^{-2})$ at $x = 1$
(f) $f(x) = \sqrt{x}(x^2 - 5x)$ at $x = 4$

9. Find the equation of the tangent to $y = (2x^3 - 4x + 2)(x^2 - 3x + 1)$ at point $(2, -10)$.

10. Find the intervals where $f(x) = (x^2 - 5x + 1)(6x - 2)$ is increasing and where it is decreasing. Round your answers to the nearest hundredth.

11. Determine the local maximum and minimum values of $g(x) = (3x^2 - 6x)(3x^2 + 6x)$, to the nearest hundredth.

12. **Communication:** Give an example to illustrate this statement: Functions f and g are increasing on an interval I, so $f \times g$ is also increasing on I.

13. **Application:** A 75-L gas tank has a leak. After t hours, the remaining volume, V, in litres is $V(t) = 75\left(1 - \frac{t}{24}\right)^2$, $0 \leq t \leq 24$. Use the product rule to determine how quickly the gas is leaking from the tank when the tank is 60% full of gas.

14. For $f(x) = (2x)(x^2 - 2x + 1)$, determine the critical numbers, the intervals where $f(x)$ is increasing or decreasing, and the extrema. Use this information to graph the function.

15. A tour operator plans to take a 60-seat bus to the next Toronto Maple Leafs home game. In the past, the trip cost \$130 per person and the bus was full. Now the tour operator wants to raise the price. She estimates that, for every \$5 she raises the price, she will lose two customers. The revenue, R, can be modelled by $R(x) = (130 + 5x)(60 - 2x)$, where x is the number of \$5 increases.

(a) What should the ticket price be to maximize revenue?

(b) Will there be any empty seats on the bus?

16. Use the product rule to show that $\frac{d}{dx}(4x^{10}) = 40x^9$, regardless of how $4x^{10}$ is factored.

17. **Check Your Understanding:** Use an example to verify this statement: The derivative of the product of two quadratic functions does not equal the product of the derivatives of the quadratics.

C 18. (a) Find the points on the graph of $y = (x^2 - 4x)(x^2 + 4x)$ at which the tangents are horizontal.

(b) What is the significance of these points?

19. **Thinking, Inquiry, Problem Solving:** Prove that $y' = u'vw + uv'w + uvw'$ if $y = uvw$ and where u, v, and w are differentiable functions of x.

20. Find the equation of a line that is perpendicular, at the point of tangency, to the tangent to $f(x) = (x^3 - 2x^2)(x + 2)$ if the tangent has a slope of -4.

ADDITIONAL ACHIEVEMENT CHART QUESTIONS

Knowledge and Understanding: Let $y = (3x + 1)(x^2 + 4)$. Find $\frac{dy}{dx}$ by

(a) applying the product rule

(b) multiplying the factors first and then differentiating

Application: The function $V(x) = x(10 - 2x)(16 - 2x)$, $0 < x < 5$, models the volume of a box.

(a) Find the extreme values of V.

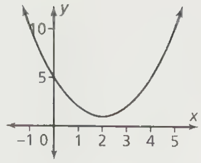
(b) Interpret the extreme values in terms of the volume of the box.

Thinking, Inquiry, Problem Solving: Suppose u and v are functions of x that are differentiable at $x = 2$. Also $u(2) = 3$, $u'(2) = -4$, $v(2) = 1$, and $v'(2) = 5$. Evaluate each derivative at $x = 2$.

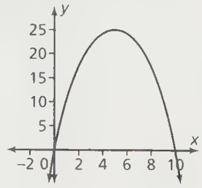
(a) $\frac{d}{dx}(uv)$ (b) $\frac{d}{dx}(3u - 2v + 2uv)$

Communication: Write the product rule in words, without using any symbols.

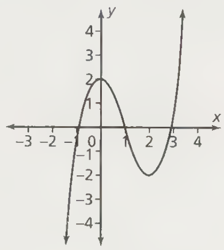
7. **(a)** $x = 2$; increasing: $x > 2$; decreasing: $x < 2$; min.: $(2, 1)$



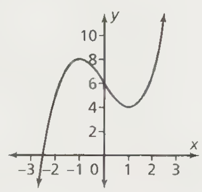
(b) $x = 5$; increasing: $x < 5$; decreasing: $x > 5$; max.: $(5, 25)$



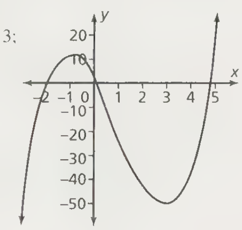
(c) $x = 0$ and $x = 2$; increasing: $x < 0, x > 2$; decreasing: $0 < x < 2$; max.: $(0, 2)$; min.: $(2, -2)$



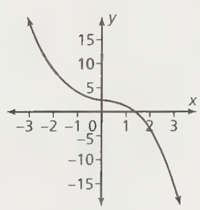
(d) $x = -1$ and $x = 1$; increasing: $x < -1$ and $x > 1$; decreasing: $-1 < x < 1$; max.: $(-1, 8)$; min.: $(1, 4)$



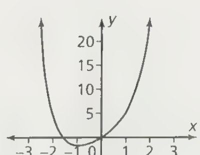
(e) $x = -1$ and $x = 3$; increasing: $x < -1, x > 3$; decreasing: $-1 < x < 3$; max.: $(-1, 13)$; min.: $(3, -51)$



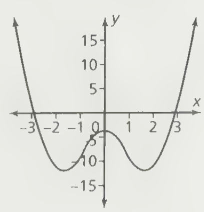
(f) $x = 0$; increasing: none; decreasing: $x \neq 0, x \in \mathbf{R}$; neither: $(0, 2)$



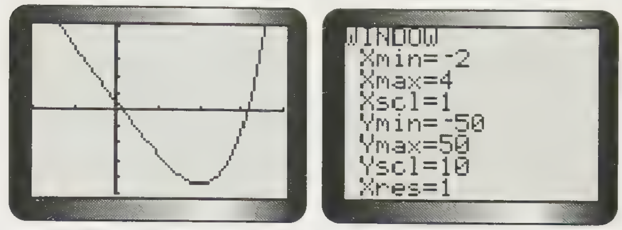
(g) $x = -1$; increasing: $x > -1$; decreasing: $x < -1$; min.: $(-1, -3)$



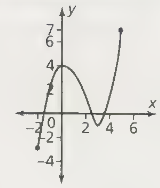
(h) $x = -\sqrt{3}, x = 0$ and $x = \sqrt{3}$; increasing: $-\sqrt{3} < x < 0, x > \sqrt{3}$; decreasing: $x < -\sqrt{3}, 0 < x < \sqrt{3}$; max.: $(0, -3)$; min.: $(-\sqrt{3}, -12)$, $(\sqrt{3}, -12)$



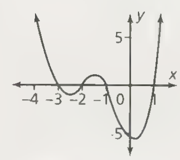
8. critical number: $x = 2$ at $(2, -44)$; increasing: $x > 2$; decreasing: $x < 2$; local min.: $(2, -44)$; local max.: none



9.



10.



11. $k = 1$

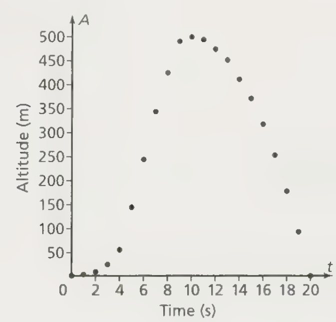
12. $k = 2$ or $k = -2$

13. 10 000 books

14. $x = 1$ cm

15. max.: day 5.5; min.: day 13.6

16.



17. no; Example: $y = x^3$ – critical point is at $x = 0$, which is neither a local min. nor a local max.

18. 15 m × 15 m

19. $(0.25, -2.625)$

20. $5/bushel

21. max. slope: none; min. slope: $-\frac{17}{3}$

4.4 Exercises, page 300

1. **(a)** $(x)(3x^2) + (1)(x^3)$

(b) $(4x)(x^2 - 2x) + (2x^2)(2x - 2)$

(c) $(3 - 2x^3)(4) + (-6x^2)(5 + 4x)$

(d) $(6x + 4)(2x^2 - 9) + (3x^2 + 4x - 6)(4x)$

(e) $(16x^3 - 16x)(4x^4 - 8x^2) + (16x^3 - 16x)(4x^4 - 8x^2)$

(f) $(4x^3 - 2x^2 + 8x)(-2x - 5) + (12x^2 - 4x + 8)(-x^2 - 5x + 3)$

2. $F'(x) = [b(x)][c'(x)] + [b'(x)][c(x)]$

3. **(a)** $105x^6 - 120x^5 + 150x^4 - 140x^3$

(b) $105x^6 - 120x^5 + 150x^4 - 140x^3$

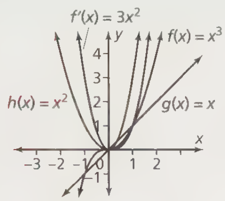
(c) same

4. (a) $-60x^3 + 33x^2 + 130x - 21$
(b) $-60x^3 + 33x^2 + 130x - 21$
(c) same

5. (a) $25x^4 + 87x^2 - 42$ (b) $18t^2 + 32t - 7$
(c) $36x^3 - 108x^2 + 72x$ (d) $7x^6 - 27x^2$
(e) $-70x^{-3} + 25x^{-6}$ (f) $\frac{9}{2}\sqrt{z^7} - 3\sqrt{z}$
(g) $25t^2 + 6t - 6t^{\frac{1}{2}}$ (h) $-\frac{5}{3}z^{-\frac{8}{3}} - \frac{4}{3}z^{-\frac{7}{3}} - 8z^{-3}$

6. $54x^5 - 60x^4 + 40x^3 - 30x^2 + 10x - 2$

7.



8. (a) 2 (b) 62 (c) 0
(d) 1080 (e) 12 (f) 5

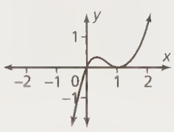
9. $10x + y - 10 = 0$

10. increasing: $x < 0.27$, $x > 3.28$; decreasing: $0.27 < x < 3.28$

11. local max.: 0.00 at (0.00, 0.00); local min.: -36.00 at $(-1.41, -36.00)$, $(1.41, -36.00)$

13. -4.84 L/h

14. critical numbers: $\frac{1}{3}$, 1;
extrema: $\left(\frac{1}{3}, \frac{8}{27}\right)$, (1, 0)
increasing: $x < \frac{1}{3}$, $x > 1$;
decreasing: $\frac{1}{3} < x < 1$



15. (a) \$140 (b) yes, 4 empty seats

18. (a) $(0, 0)$, $(2\sqrt{2}, -64)$, $(-2\sqrt{2}, -64)$ (b) critical points

20. $x - 4y - 13 = 0$, $2x - 8y - 3\sqrt{5} - 19 = 0$,
$2x - 8y + 3\sqrt{5} - 19 = 0$

4.5 Exercises, page 309

1. 35 and 35
2. 50 and 50
3. -11 and 11
4. 95 and 95
5. 110 cm × 110 cm
6. 8 m × 8 m
7. 125 m × 166.67 m
8. 450 m^2
9. 16.04 m (front) × 29.93 m (side)
10. 35.726 cm × 75.726 cm × 12.137 cm
11. 6 m × 6 m × 4 m
12. 12.1 cm × 18.2 cm × 18.2 cm
13. draw a diagram, figure out the dimension, figure out the volume V, find the derivative of V, set it equal to 0, solve for x, show V is a maximum at this x
14. 1024π cm^3
15. radius: 4.3 cm; height: 8.6 cm
16. radius: 3.84 cm; height: 7.67 cm. For an actual can, $r \doteq 4$ cm and $h \doteq 12$ cm. Reasons given for this may vary.
17. 2.74 m × 3.65 m × 3.65 m
18. 250 000 cm^3
19. 3723.37 cm^3
20. \$50
21. \$100
22. \$81.25
23. \$800
24. 12 m^2
25. 17.92 m^2
26. 42 m for the square and 33 m for the circle
27. list the relationship between variables given, set up the equation to be minimized/maximized, substitute the first equation found into the second equation found so that the equation has only one unique variable, find the derivative of the equation, set it equal to zero, solve for the variable to find the critical points, show that the critical points are a maximum, minimum, or neither
28. $2\sqrt{3}$ m^2
29. 1396 cm^3
30. radius: 8.16 cm; height: 11.55 cm

4.6 Exercises, page 319

1. (a) revenue: $-x^2 + 60x$; profit: $-x^2 + 35x - 150$;
average cost: $150x^{-1} + 25$
(b) revenue: $-0.6x^3 + 5000x$; profit: $-0.6x^3 + 2500x - 4500$;
average cost: $4500x^{-1} + 2500$
(c) revenue: $-0.025x^2 + 50x$; profit: $-0.055x^2 + 80x - 3000$;
average cost: $3000x^{-1} - 30 + 0.03x$
(d) revenue: $-0.75x^2 + 155x$; profit: $-0.8x^2 + 200x - 5350$;
average cost: $5350x^{-1} - 45 + 0.05x$

2. (a) $C'(x) = 25$; $R'(x) = 60 - 2x$; $P'(x) = -2x + 35$
(b) $C'(x) = 2500$; $R'(x) = -1.8x^2 + 5000$; $P'(x) = -1.8x^2 + 2500$
(c) $C'(x) = 0.06x - 30$; $R'(x) = -0.05x + 50$; $P'(x) = -0.11x + 80$
(d) $C'(x) = 0.1x - 45$; $R'(x) = -1.5x + 155$; $P'(x) = -1.6x + 200$

3. (a) demand: $p = \frac{3500 - x}{100}$; $0 \le x \le 3500$
(b) $p = \frac{3800 - x}{100}$; $0 \le x \le 3800$
(c) $p = \frac{2100 - x}{100}$; $0 \le x \le 2100$

4. (a) revenue: $\frac{3500x - x^2}{100}$; $R'(x) = \frac{3500 - 2x}{100}$
(b) revenue: $\frac{3800x - x^2}{300}$; $R'(x) = \frac{3800 - 2x}{300}$
(c) revenue: $\frac{2100x - x^2}{120}$; $R'(x) = \frac{2100 - 2x}{120}$

5. (a) \$2.8/L/day (b) 2236 L/day
6. (a) \$300/pizza/month (b) 4500 pizzas/month
7. (a) \$500/pair of shoes/month (b) 1100 pairs of shoes/month
8. (a) $R(x) = 600\,000x - 3000x^2$
(b) $P(x) = -4000x^2 + 400\,000x - 8\,000\,000$
(c) \$400 000 (d) \$8000/1000 organizers
(e) max. profit: \$2 000 000; sales: 50 000 units of organizers
9. (a) 22 361 units (b) 22 045 units
10. 19 704 units
11. selling x items: company earned the max. amount of profit;
selling $(x + 1)$ items: profit begins to decrease
12. 775 units
13. 30
14. (a) $x \doteq 3.21$ or $x \doteq 46.79$ (b) \$975
15. \$1100 or \$1125
16. \$70 000
17. (a) 135 units (b) 100 units
18. (a) \$160 (b) 15.79%
19. To maximize profit, the company should sell 3000 units. Therefore, the company should adjust their production capacity from 2500 to 3000.
20. 132 units
21. \$25

4.7 Exercises, page 329

1. (a) $f(x)$: parabola; $f'(x)$: slanted line with pos. slope; $f''(x)$: hor. line
(b) $f(x)$: 3 turning points; $f'(x)$: 2 turning points; $f''(x)$: 1 turning point