4.5 Finding Optimal Values for Polynomial Function Models

SETTING THE STAGE

The local concert orchestra wants to raise the price of season tickets for next year. Currently, there are 760 season-ticket holders, who paid $200 each. The marketing manager estimates that, for every $10 increase in the season-ticket price, 20 subscribers will not buy a season ticket. What season-ticket price will maximize revenue?

How often do you hear or see the terms *greatest profit, lowest cost, least time, optimum size,* and *shortest distance*? An **optimization problem** requires you to find a maximum or minimum value. In this section, you will develop a strategy for solving optimization problems.

EXAMINING THE CONCEPT

Solving Optimization Problems Involving Polynomial Functions

To learn how to deal with these problems, examine the following examples.

Example 1 Finding the Maximum Revenue

Recall the situation in Setting the Stage. What season-ticket price will maximize revenue?

Solution

Let x represent the number of $10 increases in the season-ticket price.
Let R represent the revenue. In this problem, revenue must be maximized.

$$\begin{aligned} \text{revenue} &= \text{price} \times \text{number sold} \\ \text{price} &= 200 + 10x \\ \text{number sold} &= 760 - 20x \\ R(x) &= (200 + 10x)(760 - 20x) \end{aligned}$$

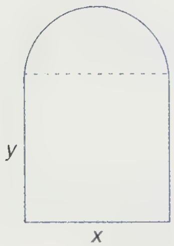
Differentiate $R(x)$ using the product rule.

$$\begin{aligned} R'(x) &= \left[\frac{d}{dx}(200 + 10x)\right](760 - 20x) + \left[\frac{d}{dx}(760 - 20x)\right](200 + 10x) \\ &= (10)(760 - 20x) + (-20)(200 + 10x) \\ &= 7600 - 200x - 4000 - 200x \\ &= 3600 - 400x \end{aligned}$$

To determine the critical numbers, let $R'(x) = 0$.

$$\begin{aligned} 3600 - 400x &= 0 \\ x &= 9 \end{aligned}$$

Use the first derivative test to verify that a maximum occurs at $x = 9$.



Norman window

25. This Norman window is made up of a semicircle and a rectangle. The total perimeter of the window is 16 m. What is the maximum area?

26. **Thinking, Inquiry, Problem Solving:** A piece of wire 75 m long is divided into two pieces. One piece is used to form a circle. The other piece is used to make a square. Find the lengths of the pieces that minimize the total area of the circle and square.

27. **Check Your Understanding:** List the steps for solving an optimization problem. Describe each step.

C 28. A picture window is in the shape of an equilateral triangle. Each side measures 4 m. Celia will glue a rectangular piece of stained glass on the window so that one side of the rectangle lies on the base of the triangle. Determine the maximum area for the piece of stained glass.

29. What is the maximum volume of a cylinder that can be inscribed in a right cone? The height of the cone is 30 cm and its base is 20 cm.

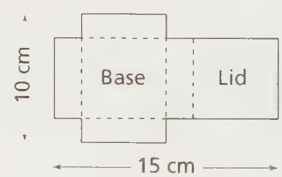
30. A cylinder is inscribed in a sphere with radius 10 cm. What are the dimensions of the cylinder if its volume is maximized?

ADDITIONAL ACHIEVEMENT CHART QUESTIONS

Knowledge and Understanding: The sum of two positive numbers is 8. Find the minimum value of the sum of the square of one number and the cube of the other number.

Application: A closed rectangular container with a square base must have a volume of 2250 m^3. The material for the top and bottom sides of the container will cost $\$2/m^2$. The material for the sides will cost $\$3/m^2$. Find the dimensions of the container so that the costs of these materials is minimized.

Thinking, Inquiry, Problem Solving: A piece of cardboard measures 10 cm by 15 cm. Two equal squares, each with sides x centimetres long, are removed from two corners of the side of the piece which measures 10 cm, as shown. Two equal rectangles are removed from the other corners so that the tabs can be folded to form a rectangular box with a lid.



(a) Write a formula for the volume, $V(x)$, of the box.

(b) Find the domain of V for this problem. Graph V over this domain.

(c) Use a graph to find the maximum volume of the box. Also find the value of x that gives the maximum volume.

(d) Confirm the maximum volume and the value of x analytically.

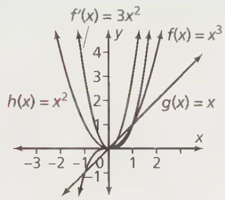
Communication: Show that the dimensions that minimize the surface area of a square-based prism with a fixed volume are equal.

4. (a) $-60x^3 + 33x^2 + 130x - 21$
(b) $-60x^3 + 33x^2 + 130x - 21$
(c) same

5. (a) $25x^4 + 87x^2 - 42$ (b) $18t^2 + 32t - 7$
(c) $36x^3 - 108x^2 + 72x$ (d) $7x^6 - 27x^2$
(e) $-70x^{-3} + 25x^{-6}$ (f) $\frac{9}{2}\sqrt{z^7} - 3\sqrt{z}$
(g) $25t^2 + 6t - 6t^{\frac{1}{2}}$ (h) $-\frac{5}{3}z^{-\frac{8}{3}} - \frac{4}{3}z^{-\frac{7}{3}} - 8z^{-3}$

6. $54x^5 - 60x^4 + 40x^3 - 30x^2 + 10x - 2$

7.



8. (a) 2 (b) 62 (c) 0
(d) 1080 (e) 12 (f) 5

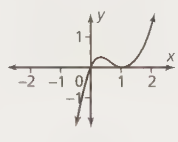
9. $10x + y - 10 = 0$

10. increasing: $x < 0.27$, $x > 3.28$; decreasing: $0.27 < x < 3.28$

11. local max.: 0.00 at (0.00, 0.00); local min.: −36.00 at (−1.41, −36.00), (1.41, −36.00)

13. −4.84 L/h

14. critical numbers: $\frac{1}{3}$, 1;
extrema: $\left(\frac{1}{3}, \frac{8}{27}\right)$, (1, 0)
increasing: $x < \frac{1}{3}$, $x > 1$;
decreasing: $\frac{1}{3} < x < 1$



15. (a) \$140 (b) yes, 4 empty seats

18. (a) $(0, 0), (2\sqrt{2}, -64), (-2\sqrt{2}, -64)$ (b) critical points

20. $x - 4y - 13 = 0$, $2x - 8y - 3\sqrt{5} - 19 = 0$,
$2x - 8y + 3\sqrt{5} - 19 = 0$

4.5 Exercises, page 309

1. 35 and 35
2. 50 and 50
3. −11 and 11
4. 95 and 95
5. 110 cm × 110 cm
6. 8 m × 8 m
7. 125 m × 166.67 m
8. 450 m^2
9. 16.04 m (front) × 29.93 m (side)
10. 35.726 cm × 75.726 cm × 12.137 cm
11. 6 m × 6 m × 4 m
12. 12.1 cm × 18.2 cm × 18.2 cm
13. draw a diagram, figure out the dimension, figure out the volume V, find the derivative of V, set it equal to 0, solve for x, show V is a maximum at this x
14. 1024π cm^3
15. radius: 4.3 cm; height: 8.6 cm
16. radius: 3.84 cm; height: 7.67 cm. For an actual can, $r \doteq 4$ cm and $h \doteq 12$ cm. Reasons given for this may vary.
17. 2.74 m × 3.65 m × 3.65 m
18. 250 000 cm^3
19. 3723.37 cm^3
20. \$50
21. \$100
22. \$81.25
23. \$800
24. 12 m^2
25. 17.92 m^2
26. 42 m for the square and 33 m for the circle
27. list the relationship between variables given, set up the equation to be minimized/maximized, substitute the first equation found into the second equation found so that the equation has only one unique variable, find the derivative of the equation, set it equal to zero, solve for the variable to find the critical points, show that the critical points are a maximum, minimum, or neither
28. $2\sqrt{3}$ m^2
29. 1396 cm^3
30. radius: 8.16 cm; height: 11.55 cm

4.6 Exercises, page 319

1. (a) revenue: $-x^2 + 60x$; profit: $-x^2 + 35x - 150$;
average cost: $150x^{-1} + 25$
(b) revenue: $-0.6x^3 + 5000x$; profit: $-0.6x^3 + 2500x - 4500$;
average cost: $4500x^{-1} + 2500$
(c) revenue: $-0.025x^2 + 50x$; profit: $-0.055x^2 + 80x - 3000$;
average cost: $3000x^{-1} - 30 + 0.03x$
(d) revenue: $-0.75x^2 + 155x$; profit: $-0.8x^2 + 200x - 5350$;
average cost: $5350x^{-1} - 45 + 0.05x$

2. (a) $C'(x) = 25$; $R'(x) = 60 - 2x$; $P'(x) = -2x + 35$
(b) $C'(x) = 2500$; $R'(x) = -1.8x^2 + 5000$; $P'(x) = -1.8x^2 + 2500$
(c) $C'(x) = 0.06x - 30$; $R'(x) = -0.05x + 50$; $P'(x) = -0.11x + 80$
(d) $C'(x) = 0.1x - 45$; $R'(x) = -1.5x + 155$; $P'(x) = -1.6x + 200$

3. (a) demand: $p = \frac{3500 - x}{100}$; $0 \le x \le 3500$
(b) $p = \frac{3800 - x}{100}$; $0 \le x \le 3800$
(c) $p = \frac{2100 - x}{100}$; $0 \le x \le 2100$

4. (a) revenue: $\frac{3500x - x^2}{100}$; $R'(x) = \frac{3500 - 2x}{100}$
(b) revenue: $\frac{3800x - x^2}{300}$; $R'(x) = \frac{3800 - 2x}{300}$
(c) revenue: $\frac{2100x - x^2}{120}$; $R'(x) = \frac{2100 - 2x}{120}$

5. (a) \$2.8/L/day (b) 2236 L/day
6. (a) \$300/pizza/month (b) 4500 pizzas/month
7. (a) \$500/pair of shoes/month (b) 1100 pairs of shoes/month
8. (a) $R(x) = 600\,000x - 3000x^2$
(b) $P(x) = -4000x^2 + 400\,000x - 8\,000\,000$
(c) \$400 000 (d) \$8000/1000 organizers
(e) max. profit: \$2 000 000; sales: 50 000 units of organizers
9. (a) 22 361 units (b) 22 045 units
10. 19 704 units
11. selling x items: company earned the max. amount of profit;
selling $(x + 1)$ items: profit begins to decrease
12. 775 units
13. 30
14. (a) $x \doteq 3.21$ or $x \doteq 46.79$ (b) \$975
15. \$1100 or \$1125
16. \$70 000
17. (a) 135 units (b) 100 units
18. (a) \$160 (b) 15.79%
19. To maximize profit, the company should sell 3000 units. Therefore, the company should adjust their production capacity from 2500 to 3000.
20. 132 units
21. \$25

4.7 Exercises, page 329

1. (a) $f(x)$: parabola; $f'(x)$: slanted line with pos. slope; $f''(x)$: hor. line
(b) $f(x)$: 3 turning points; $f'(x)$: 2 turning points; $f''(x)$: 1 turning point