5.3 Continuity of Rational Functions

SETTING THE STAGE

Explore the concepts in this lesson in more detail using Exploration 9 on page 576.

The demand for a new music CD is described by

$$D(p) = \begin{cases} \dfrac{1}{p^2} & \text{if } 0 < p \le 15 \\ 0 & \text{if } p > 15 \end{cases}$$

where D is the demand for the CD at price p in dollars.

Where is the demand function continuous?

A polynomial function is continuous for all values of the domain. In this section, you will examine how asymptotes and other features of rational functions affect the continuity of rational functions.

EXAMINING THE CONCEPT

Continuity at a Point

A function $f(x)$ is continuous at $x = a$ if and only if $\lim_{x \to a} f(x) = f(a)$. For the function to be continuous at $x = a$, the following three conditions must be true:

1. $\lim_{x \to a} f(x)$ exists

2. $f(a)$ exists (or is defined)

3. $\lim_{x \to a} f(x) = f(a)$

Example 1 Finding Points of Continuity and Discontinuity

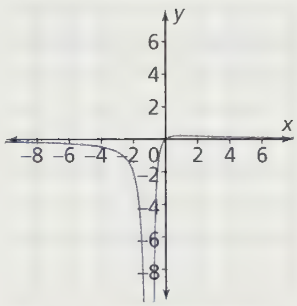
Find all numbers, $x = a$, for which each function is discontinuous. For each discontinuity, state which of the three conditions for continuity are not satisfied.

(a) $f(x) = \dfrac{x}{(x+1)^2}$ **(b)** $g(x) = \dfrac{x^2 - 9}{x - 3}$

(c) $h(x) = \begin{cases} 2x^4 - 3x^3 - x^2 + x - 1 & \text{if } x \le 2 \\ \dfrac{x^2 + 2x - 3}{x - 1} & \text{if } x > 2 \end{cases}$

Solution

(a) The function is in simplified form. The denominator, $(x + 1)^2$, is 0 when $x = -1$. The function is discontinuous at $x = -1$. The line $x = -1$ is a vertical asymptote. $f(-1)$ does not exist and $\lim_{x \to -1} f(x)$ does not exist. Conditions 1 and 2 are not satisfied.

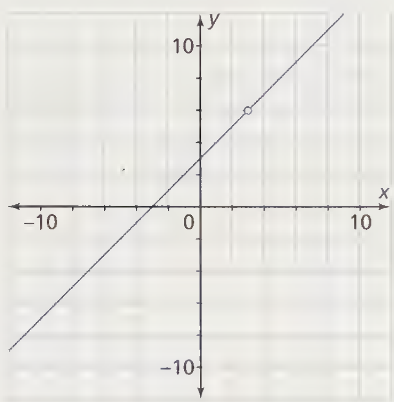


$f(x) = \frac{x}{(x+1)^2}$, discontinuous at $x = -1$

(b) First, factor.

$$\begin{aligned} g(x) &= \frac{x^2 - 9}{x - 3} \\ &= \frac{(x-3)(x+3)}{(x-3)} \\ &= x + 3 \quad \text{if } x \neq 3 \end{aligned}$$

The denominator is 0 when $x = 3$. The graph of the simplified function is the line $y = x + 3$. But the graph of the original function has a hole at $x = 3$. $g(3)$ does not exist. Condition 2 is not satisfied.

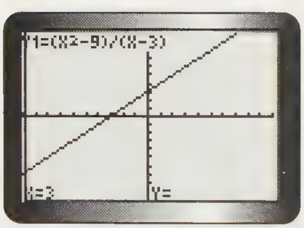


$g(x) = \frac{x^2 - 9}{x - 3}$, discontinuous at $x = 3$

(c) $h(x) = \begin{cases} 2x^4 - 3x^3 - x^2 + x - 1 & \text{if } x \leq 2 \\ \frac{x^2 + 2x - 3}{x - 1} & \text{if } x > 2 \end{cases}$

When $x < 2$, $h(x)$ is the polynomial $2x^4 - 3x^3 - x^2 + x - 1$. Since a polynomial is continuous for all values in its domain, this function is continuous for $x < 2$.

When $x > 2$, $h(x)$ is the rational function $\frac{x^2 + 2x - 3}{x - 1}$. Since a rational function is discontinuous only if the denominator equals 0, (in this case, when $x = 1$) this function is continuous for $x > 2$.



The only place where the function might be discontinuous is at $x = 2$.

The function is continuous at $x = 2$ if $\lim_{x \to 2} h(x) = h(2)$.

Since $h(2) = 2(2)^4 - 3(2)^3 - (2)^2 + (2) - 1 = 5$, $h(2)$ is defined.

And $$\begin{aligned} \lim_{x \to 2^-} h(x) &= \lim_{x \to 2^-} (2x^4 - 3x^3 - x^2 + x - 1) \\ &= 2(2)^4 - 3(2)^3 - (2)^2 + (2) - 1 \\ &= 5 \end{aligned}$$

Since $h(x) = \frac{x^2 + 2x - 3}{x - 1}$ if $x > 2$, $$\begin{aligned} \lim_{x \to 2^+} h(x) &= \lim_{x \to 2^+} \frac{x^2 + 2x - 3}{x - 1} \\ &= \frac{(2)^2 + 2(2) - 3}{(2) - 1} \\ &= 5 \end{aligned}$$

The one-sided limits are the same, so $\lim_{x \to 2} h(x) = 5$. Also $\lim_{x \to 2} h(x) = h(2)$ and $h(2) = 5$.

All three conditions for continuity are satisfied at $x = 2$. $h(x)$ is continuous for all values of its domain.

Example 2 **Examining a Jump Discontinuity**

The demand for a new music CD is described by $D(p) = \begin{cases} \frac{1}{p^2} & \text{if } 0 < p \leq 15 \\ 0 & \text{if } p > 15 \end{cases}$,

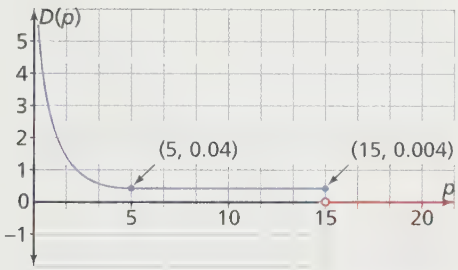
where D is the demand for the CD at price p in dollars.
Determine where the demand function is continuous.

Solution

For $0 < p < 15$, $D(p) = \frac{1}{p^2}$ is continuous for all values of p in this interval.

For $p > 15$, $D(p) = 0$. $D(p) = 0$ is continuous for all values of p such that $p > 15$.

When $p = 15$, $D(15) = \frac{1}{15^2} = \frac{1}{225}$, so $D(15)$ is defined.



$$\lim_{p \to 15^-} D(p) = \lim_{p \to 15^-} \frac{1}{p^2} = \frac{1}{225}$$

$$\lim_{p \to 15^+} D(p) = \lim_{p \to 15^+} 0 = 0$$

Since the one-sided limits are not the same, $\lim_{p \to 15} D(p)$ does not exist.

The graph of the function has a break or "jump" at $p = 15$.

The demand function has a ***jump discontinuity*** at $p = 15$.

A discontinuity is a ***jump discontinuity*** when the one-sided limits exist but are not equal. Sometimes, if a discontinuity is a hole in an otherwise continuous graph, it can be removed by redefining the function at that point.

Example 3 **Removing a Discontinuity**

Determine the point of discontinuity for $f(x) = \frac{x^2 + 3x - 10}{x - 2}$.
Then redefine the function to remove the discontinuity.

Solution

Factor the numerator.

$$f(x) = \frac{x^2 + 3x - 10}{x - 2}$$
$$= \frac{\cancel{(x - 2)}(x + 5)}{\cancel{(x - 2)}}$$
$$= x + 5, x \neq 2$$

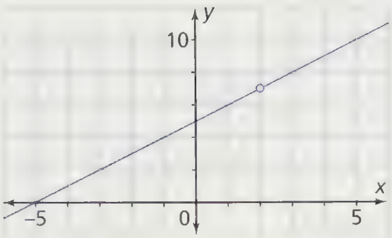
Use a table and graph $y = f(x)$.

x	$f(x)$
−3	2
−2	3
−1	4
0	5
1	6
2	undefined
3	8
4	9

The function has a discontinuity at $x = 2$. The discontinuity is represented by a hole in the graph. $f(2)$ is not defined.

However, as x approaches 2 from either side, $f(x)$ approaches 7.

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = 7$, so $\lim_{x \to 2} f(x) = 7$



The function will satisfy the conditions for continuity at $x = 2$ if the function is defined so that $f(2) = 7$.

Define $f(x) = \begin{cases} \dfrac{x^2 + 3x - 10}{x - 2} & \text{if } x \neq 2 \\ 7 & \text{if } x = 2 \end{cases}$ to remove the discontinuity.

The Continuity of Rational Functions

Rational functions are continuous at every point where they are defined, so they are continuous on their domain. However, rational functions are undefined at the zeros of their denominator. At these points they have either removable discontinuities (holes) or vertical asymptotes.

Example 4 Determining Continuous Functions

Which functions in examples 1, 2, and 3 are continuous?

Solution

In Example 1, the functions in parts (a) and (b) are discontinuous because each one has a discontinuity. The function in (c) is continuous since it is continuous at every point in its domain.

In Example 2, the demand function is not continuous since it has a discontinuity at $p = 15$.

In Example 3, the original function is not continuous since it has a discontinuity, but the redefined function is continuous since it is continuous at every point in its domain.

CHECK, CONSOLIDATE, COMMUNICATE

1. Describe, with examples, where a rational function can be discontinuous.
2. What is a removable discontinuity, and how is it removed?
3. Give three examples of continuous rational functions and three examples of functions that are not continuous. Explain why the latter three are not continuous.

KEY IDEAS

- A rational function $h(x) = \dfrac{f(x)}{g(x)}$ is continuous at $x = a$ if $g(a) \neq 0$.
- A rational function in simplified form has vertical asymptotes at the zeros of the denominator.

- The graph of a rational function will have a hole at $x = a$ if $(x - a)$ is a common factor of the functions in the numerator and denominator.
- A discontinuity is a ***jump discontinuity*** when the one-sided limits exist but are not equal.
- A removable discontinuity can be removed by redefining the function at the point of discontinuity to make the function continuous.
- Rational functions are continuous on their domain, but may have points of discontinuity at the zeros of their denominators that result in holes or vertical asymptotes.

5.3 Exercises

A

1. Each function has a discontinuity at $x = 2$. In each case, state which of the continuity conditions are not satisfied.

(a) $f(x) = \dfrac{x^2 + x - 6}{x - 2}$ **(b)** $f(x) = \begin{cases} \dfrac{x^2 + x - 6}{x - 2} & \text{if } x \neq 2 \\ 2 & \text{if } x = 2 \end{cases}$

(c) $f(x) = \dfrac{x^2 - x + 6}{x - 2}$

2. For each function, find any points of discontinuity. State which of the conditions of continuity are not satisfied.

(a) $f(x) = \dfrac{1}{(x + 3)^2}$ **(b)** $f(x) = \dfrac{x + 1}{x^2 - 4x - 5}$ **(c)** $f(x) = \dfrac{1}{x^2 - 1}$

3. Sketch each function. Determine any discontinuities.

(a) $f(x) = \dfrac{2}{x - 3}$ **(b)** $f(x) = 2x - \dfrac{1}{x + 1}$ **(c)** $f(x) = \dfrac{x^2 - 9}{x + 3}$

(d) $f(x) = \dfrac{4x^2 - x^4}{x^2 - 4}$ **(e)** $f(x) = \dfrac{x + 2}{x^2 - 3x - 10}$ **(f)** $f(x) = \dfrac{x^2 - x - 6}{x - 3}$

4. Determine whether each function is continuous at $x = -1$ or not. Justify your decision.

(a) $f(x) = \begin{cases} \dfrac{x^2 - 1}{x + 1} & \text{if } x \neq -1 \\ -2 & \text{if } x = -1 \end{cases}$ **(b)** $f(x) = \begin{cases} \dfrac{1}{x + 1} & \text{if } x \neq -1 \\ 0 & \text{if } x = -1 \end{cases}$

5. **Knowledge and Understanding:** Determine where each function is continuous.

(a) $f(x) = \dfrac{3x}{x - 4}$ **(b)** $g(x) = \dfrac{x^2 - x - 12}{x^2 - 16}$ **(c)** $h(x) = \begin{cases} \dfrac{1}{x - 3} & \text{if } x < 2 \\ x - 4 & \text{if } x \geq 2 \end{cases}$

B

6. **Communication:** Explain the difference between saying that $f(x)$ is continuous at $x = a$ and saying that $\lim_{x \to a} f(x)$ exists.

7. Determine where each function is continuous.

   (a) $f(x) = \dfrac{x}{x^2 + x}$ **(b)** $f(x) = \dfrac{x}{x^2 - 1}$ **(c)** $f(x) = \dfrac{x - 4}{x^2 - 16}$

8. Show that $f(x) = \dfrac{x^2 + 3x - 4}{x + 4}$ has a discontinuity at $x = -4$.

   How must $f(-4)$ be defined to make $f(x)$ continuous?

9. Define $f(1)$ so that $f(x) = \dfrac{x^2 - 1}{x - 1}$ is continuous for all values of x.

10. Where possible, define each function so that it is continuous for all values of x.

    (a) $f(x) = \dfrac{x^2 - 9}{x - 3}$ **(b)** $f(x) = \dfrac{x - 2}{x^2 - 4}$

11. **Application:** For what values of a does $f(x) = \dfrac{x^2 + x + a}{x^2 + 2x - 3}$ have a removable discontinuity? For each value of a, redefine the function to make it continuous.

12. For $f(x) = \begin{cases} \dfrac{2}{x^2 + 1} & \text{if } x < 0 \text{ or } x > 1 \\ x & \text{if } 0 \le x \le 1 \end{cases}$

    (a) sketch $f(x)$

    (b) determine where the function is continuous

13. Determine where each function is continuous.

    (a) $f(x) = \dfrac{1}{x^2}$ **(b)** $f(x) = \dfrac{x}{x^2 + 1}$ **(c)** $f(x) = \dfrac{x^2 - 4}{x - 2}$

    (d) $f(x) = \dfrac{x - 1}{1 - x}$ **(e)** $f(x) = \dfrac{x + 1}{x^2 - 4x - 5}$

    (f) $f(x) = \begin{cases} \dfrac{x}{4} & \text{if } x \le 2 \\ \dfrac{1}{x} & \text{if } x > 2 \end{cases}$ **(g)** $f(x) = \begin{cases} \dfrac{x^2 + 2x - 3}{x^2 - 1} & \text{if } x \ne \pm 1 \\ 2 & \text{if } x = 1 \end{cases}$

14. **Thinking, Inquiry, Problem Solving:** Define a function that has a removable discontinuity at $x = -2$ and a jump discontinuity at $x = 3$.

15. Which functions in questions 12, 13, and 14 are continuous? Give reasons for your answers.

16. Let $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \dfrac{A}{x + 3} & \text{if } x \ge 1 \end{cases}$. Find A if the function is continuous at $x = 1$.

17. **Check Your Understanding:** What are the conditions for continuity at a point, and at what points are rational functions continuous? Explain, with an example, the steps for finding the points of discontinuity for a rational function.

C **18.** What must be true about A and B for $f(x) = \begin{cases} \frac{Ax - B}{x - 2} & \text{if } x \leq 1 \\ 3x & \text{if } 1 < x < 2 \\ Bx^2 - A & \text{if } x \geq 2 \end{cases}$

if the function is continuous at $x = 1$ but discontinuous at $x = 2$?

19. Use graphs to show whether the following is true or not:

A function that is continuous at every x-value on the interval $a < x < b$ will take on every value between $f(a)$ and $f(b)$ on $a < x < b$. But a function that is not continuous on $a < x < b$ might not take on every value between $f(a)$ and $f(b)$ in $a < x < b$.

ADDITIONAL ACHIEVEMENT CHART QUESTIONS

Knowledge and Understanding: Find the points of discontinuity, if any.

(a) $f(x) = \frac{x}{x^2 + 1}$

(b) $f(x) = \frac{x - 4}{x^2 - 16}$

(c) $f(x) = \begin{cases} 2x + 3 & \text{if } x \leq 4 \\ 7 + \frac{16}{x} & \text{if } x > 4 \end{cases}$

(d) $f(x) = \begin{cases} \frac{3}{x - 1} & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$

Application: A student parking lot at a university charges \$2 for the first half hour (or any part) and \$1 for each subsequent half hour (or any part) up to a daily maximum of \$10.

(a) Graph cost as a function of the time parked.

(b) Using a graph, discuss the meaning of discontinuities to a student who parks in the student parking lot.

Thinking, Inquiry, Problem Solving: Create the graph of a function so that it is not continuous at $x = 3$ and so that the function becomes continuous at $x = 3$ if its value at $x = 3$ is changed from $f(3) = 1$ to $f(3) = 0$.

Communication: Determine whether each function is continuous or not. Explain your reasoning.

(a) your exact height as a function of time

(b) the cost of a taxi ride in your city as a function of the distance travelled

(c) the volume of a melting ice cube as a function of time

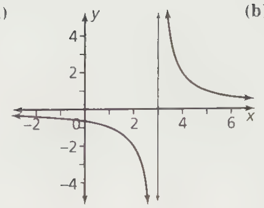
(d) the Earth's population as a function of time

5.2 Exercises, page 367

1. (a) $\frac{4}{[illegible]}$ (b) -2 (c) [illegible]
   (d) $\frac{33}{10}$ (e) 0 (f) $\frac{3}{5}$
3. does not exist; $\lim_{x \to 3^-} f(x) = -\infty$ and $\lim_{x \to 3^+} f(x) = \infty$
4. does not exist; $\lim_{x \to 3^-} g(x) = \infty$ and $\lim_{x \to 3^+} g(x) = \infty$
5. (a) ∞ (b) $-\infty$ (c) ∞
   (d) $-\infty$ (e) ∞ (f) $-\infty$
6. (a) undefined (b) indeterminate
   (c) indeterminate (d) cannot substitute infinity
7. (a) 0.25 (b) 2 (c) 3 (d) -3
   (e) $\frac{1}{3}$ (f) 12 (g) $-\frac{3}{5}$ (h) $-\frac{1}{25}$
   (i) 8 (j) 0 (k) $\frac{1}{2}$ (l) 27
8. (a) asymp.: $x = -2$; $\lim_{x \to -2^-} f(x) = \infty$, $\lim_{x \to -2^+} f(x) = -\infty$
   (b) asymp.: $x = -3, 3$; $\lim_{x \to -3^-} f(x) = -\infty$, $\lim_{x \to -3^+} f(x) = \infty$,
   $\lim_{x \to 3^-} f(x) = \infty$, $\lim_{x \to 3^+} f(x) = -\infty$
   (c) asymp.: $x = -3$; $\lim_{x \to -3^-} f(x) = -\infty$, $\lim_{x \to -3^+} f(x) = \infty$
   (d) asymp.: $x = -\frac{2}{3}, 5$; $\lim_{x \to -\frac{2}{3}^-} f(x) = \infty$, $\lim_{x \to -\frac{2}{3}^+} f(x) = -\infty$,
   $\lim_{x \to 5^-} f(x) = \infty$, $\lim_{x \to 5^+} f(x) = -\infty$
9. graph function, take limit as $x \to \infty$ by finding an end-behavior model, take limit as $x \to \infty$ by dividing numerator and denominator by highest power of x in denominator
10. (a) $\frac{5}{4}$ (b) $\frac{4}{3}$ (c) -3 (d) 0
    (e) ∞ (f) 1 (g) 1 (h) ∞
11. Vitaly did not simplify to reduce the denominator.
12. (a) -1 (b) does not exist; $f(3)$ does not exist
    (c) -5 (d) $\frac{3}{5}$
13. (a) vert. asymp.: $x = -5$; $\lim_{x \to -5^-} f(x) = \infty$, $\lim_{x \to -5^+} f(x) = -\infty$;
    hor. asymp.: $y = 3$
    (b) vert. asymp.: $x = 1$; $\lim_{x \to 1^-} g(x) = \infty$, $\lim_{x \to 1^+} g(x) = \infty$;
    hor. asymp.: $y = 1$
    (c) vert. asymp.: $x = -2$; $\lim_{x \to -2^-} h(x) = -\infty$, $\lim_{x \to -2^+} h(x) = \infty$;
    hor. asymp.: $y = 1$
    (d) vert. asymp.: $x = 2$; $\lim_{x \to 2^-} m(x) = -\infty$, $\lim_{x \to 2^+} m(x) = \infty$;
    hor. asymp.: none
14. (a) no (b) yes
15. (a) \$9080 (b) \$4085.56
    (c) car's value approaches \$90 (d) \$90
17. (a) $y = 0$ (b) $y = 4$
18. can say $\lim_{x \to a} (f(x) \times g(x)) \neq \lim_{x \to a} f(x) \times \lim_{x \to a} g(x)$
19. fails for $\lim_{x \to a} \frac{p(x)}{q(x)}$ when $q(a) = 0$; Examples will vary.
    $f(x) = \lim_{x \to 1} \frac{x^2 - 2}{x - 1}$
20. $7.99 \le f(x) \le 8.01$
21. (a) $x = 3$ (b) $2.99 \le x \le 3.01$
    (c) $2.999\,000\,5 \le x \le 3.001\,000\,5$
22. limit is in indeterminant form $\left(\frac{0}{0}\right)$ so may or may not exist; Examples will vary.

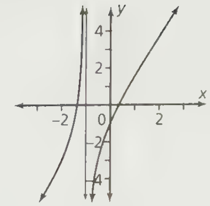
5.3 Exercises, page 375

1. (a) condition 2 (b) condition 3
   (c) conditions 1 and 2
2. (a) $x = -3$, conditions 1 and 2
   (b) $x = -1$, condition 2; $x = 5$, conditions 1 and 2
   (c) $x = 1$, 1, conditions 1 and 2
3. (a)



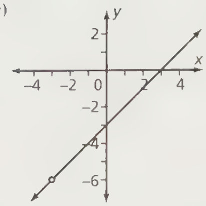
$x = 3$

(b)



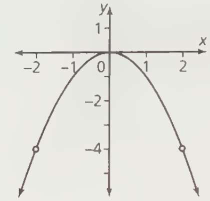
$x = -1$

(c)



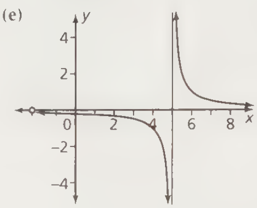
$x = 3$

(d)



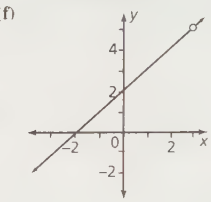
$x = -2$ and $x = 2$

(e)



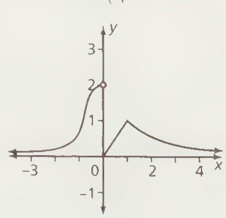
$x = -2$ and $x = 5$

(f)



$x = 3$

4. (a) continuous (b) not continuous
5. (a) all points except $x = 4$
   (b) all points except $x = -4$ and $x = 4$
   (c) all points
6. continuous at $x = a$: function satisfies all 3 conditions for continuity; $\lim_{x \to a} f(x)$ exists: function approaches a value at $x = a$ but does exist at $x = a$
7. (a) all points except $x = -1$ and $x = 0$
   (b) all points except $x = -1$ and $x = 1$
   (c) all points except $x = -4$ and $x = 4$
8. $f(-4) = 5$
9. $f(1) = 2$
10. (a) $f(x) = \begin{cases} \frac{x^2 - 9}{x - 3}, & x \neq 3 \\ 6, & x = 3 \end{cases}$ (b) not possible
11. $a = 6$: $f(x) = \begin{cases} \frac{x^2 + x - 6}{x^2 + 2x - 3}, & x \neq 3 \\ \frac{5}{4}, & x = 3 \end{cases}$.
    $a = -2$: $f(x) = \begin{cases} \frac{x^2 + x - 2}{x^2 + 2x - 3}, & x \neq 1 \\ \frac{3}{4}, & x = 1 \end{cases}$
12. (a)



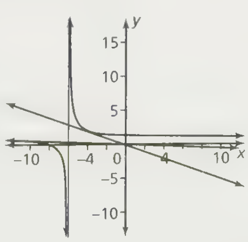
    (b) all points except $x = 0$
13. (a) all points except $x = 0$ (b) at all points
    (c) all points except $x = 2$ (d) all points except $x = 1$

(e) all points except $x = -1$ and $x = 5$
(f) at all points
(g) all points except $x = -1$

14. Answers will vary. Example: $f(x)=\begin{cases}\dfrac{x^2+x-2}{x+2}, & x\le 3\\ x+2, & x>3\end{cases}$
15. functions in 13(b) and 13(f); satisfy all 3 conditions of continuity
16. $A = 4$
17. $\lim_{x \to a} f(x)$ exists, $f(a)$ exists, $\lim_{x \to a} f(x) = f(a)$; Examples will vary.
18. $A = B - 3$ and either $B > 1$ and $A > -2$ or $B < 1$ and $A < -2$
19. statement is true; Examples will vary.

5.4 Exercises, page 383

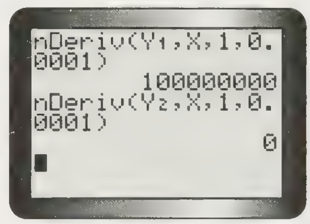
1. (a) $\frac{3}{(3-x)^2}$ (b) $\frac{3}{(1+x)^2}$ (c) $\frac{3x^2+2}{x^2}$
2. (a) $\frac{6}{(x+3)^2}$ (b) $\frac{6x^5-72x^3}{(x^2-6)^2}$
(c) $\frac{x^4+2x^3+5x^2-2}{(x^2+x+1)^2}$ (d) $\frac{2x^2+20x+30}{x^2(x+3)^2}$
(e) $\frac{3x^2-2x^3}{(1-x)^2}$ (f) $\frac{bc-ad}{(cx-d)^2}$
(g) $\frac{2x^2+6x+2}{(2x+3)^2}$ (h) $\frac{15x^4+40x^3-9}{(x+2)^2}$
(i) $\frac{2x^4-2}{x^3}$ (j) $\frac{-5x^2+2x-3}{x^2(x-3)^2}$
4. (a) $\frac{-x-11}{(x+1)^3}$ (b) $\frac{10x^3-45x^2}{2(x-3)^2}$
(c) $\frac{x^2-4x+10}{(x-2)^2}$ (d) $\frac{9x^2+90x-17}{(3x+1)^2(3x-2)^2}$
5. $\frac{3x^4+2}{x^2}$
6. $y = 3x - 2$
7. (a) $y = \frac{3}{4}x + \frac{25}{4}$ (b) $y = -\frac{3}{4}x - \frac{5}{4}$
8. (a) $\left(1, \frac{5}{2}\right)$ and $\left(-1, -\frac{5}{2}\right)$ (b) $\left(2, \frac{1}{2}\right)$ and $\left(-2, \frac{3}{2}\right)$
9. (a) position: 1, velocity: $\frac{1}{6}$, acceleration: $-\frac{1}{18}$, speed: $\frac{1}{6}$
(b) position: $\frac{8}{3}$, velocity: $\frac{4}{9}$, acceleration: $\frac{10}{27}$, speed: $\frac{4}{9}$
10. yes, when $t = 0.24$ s
11. $t = 2.83$
12. $c'(t) = \frac{250}{(25+t)^2}$
13. $t = 1.87$ h
14. $p'(t) = \frac{25}{(t+1)^2}$
15. $\frac{dy}{dx} = \frac{6x}{(2x^2+1)^2}$, positive for $x > 0$
16. $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
17. (a) 1 cm (b) $t = 1$ s (c) 0.25 cm/s
(d) no; radius will never reach 2 cm ($y = 2$ is horizontal asymptote of the graph)
18. $P'(t) = -\frac{390}{(3t+2)^2}$ is rate of change of population. $P''(t) = \frac{2340}{(3t+2)^3}$ is how the rate of change of population is changing. Examples will vary. As years pass, rate of change is increasing.
19. $y = -\frac{x}{2}$ and $y = -\frac{x}{18}$



20. Examples will vary. Rule: If $f(x) = ax + b$ and $y = \frac{1}{f(x)}$, then $y' = \frac{-a}{(ax+b)^2}$.

5.5 Exercises, page 391

1. (a) $\lim_{h \to 0^-} \frac{f(2+h)-f(2)}{h} \neq \lim_{h \to 0^+} \frac{f(2+h)-f(2)}{h}$
(b) corner at $x = 2$, so $\lim_{h \to 0^-} \frac{f(2+h)-f(2)}{h} \neq \lim_{h \to 0^+} \frac{f(2+h)-f(2)}{h}$
(c) vertical tangent at (2, 1), so slope is undefined
2. (a) **i.** all points except $x = 0$; **ii.** nowhere; **iii.** $x = 0$
(b) **i.** all points except $x = 0$ and $x = 3$; **ii.** $x = 3$; **iii.** $x = 0$
(c) **i.** all points except $x = 1$; **ii.** $x = 1$; **iii.** nowhere
3. (a) discontinuity (b) corner (c) vertical tangent
(d) cusp (e) discontinuity (f) vertical tangent
4. (a) not defined (vertical asymptote)
(b) not defined (vertical asymptote)
(c) cusp (d) corner (e) not defined (vertical asymptote)
(f) restricted domain (g) vertical asymptote
(h) discontinuity
5. (c) cusp at (2, 0) but is differentiable at $x = 1$
6. (a) all points except $x = -1$ and $x = 4$
(b) all points except $x = 2$ (c) all points except $x = 1$
7. (a) $x = 0.25$ and $x = 0$ (b) $x = -3$ and $x = 3$
(c) $1 < x < 6$
8. left and right hand limit of $\frac{f(x+h)-f(x)}{h}$ would be different
9. (a) not defined at $x = 1$ (vertical asymptote)
(b)



(c) $f(x)$: 100 000 000; $g(x)$: 0 (d) Answers will vary.
10. no
12. discontinuity or restricted domain, graph has corner or cusp, graph has vertical tangent; Examples will vary.
13. (a) $a = 4 + 2b$ (b) $a = -8$, $b = -6$
14. (a) $6b = 2a + 5$ (b) $a = \frac{11}{4}$, $b = \frac{7}{4}$
15. converse is not true. Counterexamples will vary.

Exercise 5.6, page 401

1. 1.5, min.
2. 6 cm × 6 cm
3. 3.16 m × 6.33 m
4. 8 m × 12 m; 48 m
5. 17.3 m × 34.6 m
6. 25 m × 40 m
7. 60 m × 80 m
8. 50 m × 75 m
9. 4 m × 8 m
10. 20 km/h
11. 11.3 cm × 17.0 cm
12. 8.5 cm × 14.1 cm
13. 5 cm × 10 cm × 10 cm
14. 2.38 m × 2.38 m × 2.65 m
15. 14.94 cm × 14.94 cm × 22.40 cm
16. 0.585 m × 1.170 m × 0.438 m
17. radius 2.75 m, height 21.05 m
18. radius 0.66 m, height 1.54 m
19. (a) 0.60 m × 1.29 m × 1.32 m
(b) 0.50 m × 1.41 m × 1.41 m