5.4 Rate of Change of a Rational Function—The Quotient Rule

SETTING THE STAGE

Polluted water flows at a rate of 3 m^3/min into a pond. The pond initially holds 10 000 m^3 of unpolluted water. The concentration of pollutant in the polluted water is 9 kg/m^3. The concentration of pollutant, c, in the pond at t minutes is modelled by $c(t) = \frac{27t}{10\,000 + 3t}$, where c is measured in kilograms per cubic metre. What is the domain of this function? At what rate is the concentration changing after one hour? one day? one week? What is the average rate of change of $c(t)$ in the first week? What will happen to the concentration of pollutant in the long run?

In this problem, the rate at which the concentration of pollutant is changing at time t is given by the derivative $c'(t)$. In this section, you will develop techniques that will enable you to determine the derivative of a rational function.

EXAMINING THE CONCEPT

The Derivative of a Rational Function

Recall that the derivative of $f(x)$ is $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ for all x for which the limit exists.

Example 1 **Finding the Derivative of a Rational Function from First Principles**

For $c(t) = \frac{27t}{10\,000 + 3t}$, find $c'(t)$ from first principles.

Solution

$$c'(t) = \lim_{h \to 0} \frac{c(t+h) - c(t)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{27(t+h)}{10\,000 + 3(t+h)} - \frac{27t}{10\,000 + 3t}\right] \quad \text{Find a common denominator.}$$

It is important to note that the derivative of a quotient of two differentiable functions is not the quotient of the derivatives.

$$= \lim_{h \to 0} \frac{1}{h}\left\{\frac{27(t+h)(10\,000 + 3t) - 27t[10\,000 + 3(t+h)]}{[10\,000 + 3(t+h)](10\,000 + 3t)}\right\} \quad \text{Expand the numerator.}$$

$$= \lim_{h \to 0} \frac{27}{h}\left\{\frac{10\,000t + 3t^2 + 10\,000h + 3ht - (10\,000t + 3t^2 + 3ht)}{[10\,000 + 3(t+h)](10\,000 + 3t)}\right\} \quad \text{Simplify.}$$

$$= \lim_{h \to 0} \frac{27}{h}\left\{\frac{10\,000h}{[10\,000 + 3(t+h)](10\,000 + 3t)}\right\} \quad \text{Simplify.}$$

$$= \lim_{h \to 0} \frac{270\,000}{[10\,000 + 3(t+h)](10\,000 + 3t)} \quad \text{Evaluate the limit.}$$

$$= \frac{270\,000}{(10\,000 + 3t)^2}$$

There is a simpler way of finding the derivative for a rational function.

The Quotient Rule for Derivatives

Let $h(x) = \frac{f(x)}{g(x)}$. If both $f'(x)$ and $g'(x)$ exist, the derivative of $h(x)$ is

$$h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}, \text{ where } g(x) \neq 0.$$

In Leibniz notation, $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\left[\frac{d}{dx}f(x)\right]g(x) - \left[\frac{d}{dx}g(x)\right]f(x)}{[g(x)]^2}$, $g(x) \neq 0$.

The rule in words: The derivative of the top times the bottom minus the derivative of the bottom times the top all over the bottom squared.

Proof

The rule for finding the derivative of the quotient of two functions follows from the product rule for derivatives. Suppose that there are functions f and g, and that $g(x) \neq 0$. Then, $\frac{f(x)}{g(x)}$ defines a quotient of the two functions.

Let $h(x) = \frac{f(x)}{g(x)}$ — Multiply both sides by $g(x)$.

$g(x)h(x) = f(x)$ — Differentiate both sides with respect to x.

$g'(x)h(x) + h'(x)g(x) = f'(x)$ — Solve for $h'(x)$.

$h'(x) = \frac{f'(x) - g'(x)h(x)}{g(x)}$ — Substitute $h(x) = \frac{f(x)}{g(x)}$.

$= \frac{f'(x) - g'(x)\frac{f(x)}{g(x)}}{g(x)}$ — Multiply both the numerator and the denominator by $g(x)$.

$h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$

Example 2

Using the Quotient Rule

Find the derivative of each rational function using the quotient rule. Verify with graphing technology.

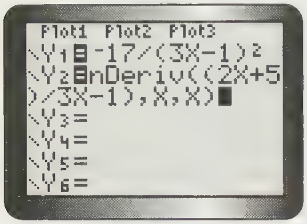
(a) $y = \frac{2x + 5}{3x - 1}$ (b) $y = \frac{x^3 - 3}{1 + 4x^2}$ (c) $y = \frac{x^2}{(x + 2)(x - 3)}$

Technology Help: For help with using the numerical derivative operation, nDeriv(, see page 595 of the Technology Appendix.

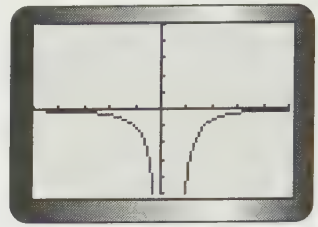
Solution

Use the quotient rule to find the derivative. To verify, graph the derivative function you found with the TI-83 Plus by entering the function into **Y1** of the equation editor. Then enter the numerical derivative of the original function as **Y2**. Both functions should yield the same graph.

(a) Apply the quotient rule with $f(x) = 2x + 5$ and $g(x) = 3x - 1$.

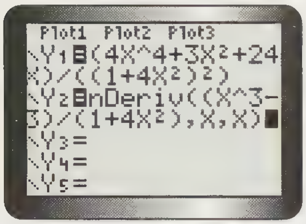


$$\begin{aligned}\frac{dy}{dx} &= \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2} \\ &= \frac{2(3x-1) - (3)(2x+5)}{(3x-1)^2} \\ &= \frac{6x - 2 - 6x - 15}{(3x-1)^2} \\ &= \frac{-17}{(3x-1)^2}\end{aligned}$$

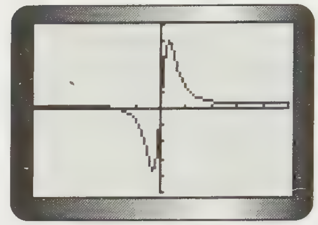


−4.7 ≤ x ≤ 4.7; −5 ≤ y ≤ 5

(b) $f(x) = x^3 - 3$ and $g(x) = 1 + 4x^2$

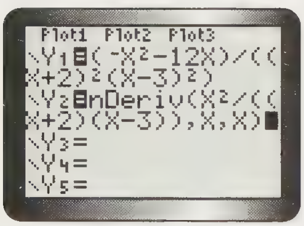


$$\begin{aligned}\frac{dy}{dx} &= \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2} \\ &= \frac{(3x^2)(1+4x^2) - (x^3-3)(8x)}{(1+4x^2)^2} \\ &= \frac{3x^2 + 12x^4 - 8x^4 + 24x}{(1+4x^2)^2} \\ &= \frac{4x^4 + 3x^2 + 24x}{(1+4x^2)^2} \\ &= \frac{x(4x^3 + 3x + 24)}{(1+4x^2)^2}\end{aligned}$$

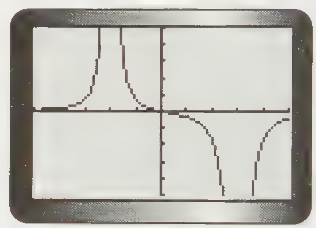


−4.7 ≤ x ≤ 4.7; −5 ≤ y ≤ 5

(c) Here $f(x) = x^2$ and $g(x) = (x + 2)(x - 3)$. Use the product rule to find $g'(x)$.



$$\begin{aligned}\frac{dy}{dx} &= \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2} \\ &= \frac{(2x)(x+2)(x-3) - [(1)(x-3) + (x+2)(1)]x^2}{[(x+2)(x-3)]^2} \\ &= \frac{(2x)(x^2 - x - 6) - x^2(2x-1)}{(x+2)^2(x-3)^2} \\ &= \frac{2x^3 - 2x^2 - 12x - 2x^3 + x^2}{(x+2)^2(x-3)^2} \\ &= \frac{-x^2 - 12x}{(x+2)^2(x-3)^2} \\ &= \frac{-x(x+12)}{(x+2)^2(x-3)^2}\end{aligned}$$



−4.7 ≤ x ≤ 4.7; −5 ≤ y ≤ 5

For most functions that are quotients, the derivative function is also a quotient. Use the quotient rule again to find the second derivative.

Example 3 **Finding the Second Derivative of a Rational Function**

An object moves along a straight line. The object's position, s, at t seconds is modelled by $s(t) = \frac{5t}{t^2 + 1}$. When does the object change direction? What is its acceleration at that instant?

Solution

When the object changes direction, its velocity, $s'(t)$, changes sign.

The velocity function is $v(t) = s'(t)$.

$$s'(t) = \frac{(5)(t^2 + 1) - (2t)(5t)}{(t^2 + 1)^2}$$
$$= \frac{5 - 5t^2}{t^4 + 2t^2 + 1} \text{ or } \frac{5(1 - t)(1 + t)}{t^4 + 2t^2 + 1}$$

$s'(t) = 0$ when $t = \pm 1$. But $t \geq 0$, so the negative root is inadmissible.

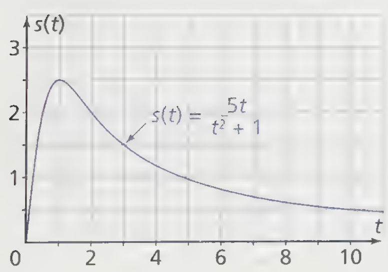
The velocity changes sign when $t = 1$. The object changes direction after exactly one second.

The acceleration function is $a(t) = v'(t) = s''(t)$.

$$s''(t) = \frac{-10t(t^4 + 2t^2 + 1) - (4t^3 + 4t)(5 - 5t^2)}{(t^4 + 2t^2 + 1)^2}$$
$$= \frac{-10t^5 - 20t^3 - 10t - (20t^3 - 20t^5 + 20t - 20t^3)}{[(t^2 + 1)^2]^2} \quad \text{Simplify.}$$
$$= \frac{10t^5 - 20t^3 - 30t}{(t^2 + 1)^4} \quad \text{Factor.}$$
$$= \frac{10t(t^4 - 2t^2 - 3)}{(t^2 + 1)^4}$$
$$= \frac{10t(t^2 - 3)(t^2 + 1)}{(t^2 + 1)^4} \quad \text{Simplify.}$$
$$= \frac{10t(t^2 - 3)}{(t^2 + 1)^3}$$

Therefore, $s''(1) = \frac{10(1 - 3)}{(2)^3}$, or -2.5.

The object's acceleration at the instant it changes direction is -2.5 units/s^2.



Graph the position function. When $t = 1$, the object has stopped briefly. Before $t = 1$, the object was moving away from a point. After $t = 1$, the object is moving toward the point. Velocity is represented by the slopes of tangent lines to this graph. At the maximum point on the graph, the slope of the tangent line is 0. The velocity is decreasing before $t = 1$. And the velocity is decreasing after $t = 1$. The graph is concave down at its peak. The acceleration is negative.

Example 4

Analyzing the Pollution Problem

Recall the original problem in Setting the Stage:

Polluted water flows at a rate of 3 m^3/min into a pond. The pond initially holds 10 000 m^3 of unpolluted water. The initial concentration of pollutant in the polluted water is 9 kg/m^3. The concentration of pollutant, c, in the pond at t minutes is modelled by $c(t) = \frac{27t}{10\,000 + 3t}$.

(a) What is the domain of this function?

(b) At what rate is the concentration changing after one hour? one day? one week?

(c) What is the average rate of change of $c(t)$ in the first week?

(d) What will happen to the concentration of pollutant in the long run?

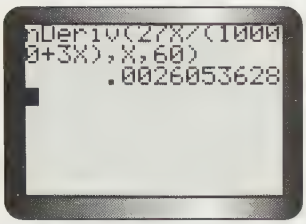
Solution

(a) Since t represents time, the domain of $c(t) = \frac{27t}{10\,000 + 3t}$ is restricted to all real numbers greater than or equal to 0, $\{t \mid t \geq 0, t \in \mathbf{R}\}$.

(b) Find the derivative.

$$c'(t) = \frac{27(10\,000 + 3t) - (3)(27t)}{(10\,000 + 3t)^2} \qquad \text{Simplify.}$$

$$= \frac{270\,000}{(10\,000 + 3t)^2}$$



You could find these values more quickly using **nDeriv(**, which is accurate to five decimal places.

At 1 h, $t = 60$, and $c'(60) \doteq 0.0026$.

At 1 h, the concentration is increasing at about 0.0026 kg/m^3/min.

After one day, $t = 1440$, and $c'(1440) \doteq 0.0013$.

After one day, the concentration is increasing at about 0.0013 kg/m^3/min.

After one week, $t = 10\,080$, and $c'(10\,080) \doteq 0.0002$.

After one week, the concentration is increasing at about 0.0002 kg/m^3/min. The rate at which the concentration is increasing decreases over time.

(c) The average rate of change of $c(t)$ in the first week is $\frac{c(10\,080) - c(0)}{10\,080 - 0}$, or about 0.0007 kg/m^3/min.

(d) To determine what happens to the concentration of pollutant in the long run, find $\lim_{t \to \infty} c(t)$.

$$\lim_{t \to \infty} \frac{27t}{10\,000 + 3t} = \lim_{t \to \infty} \frac{\frac{27t}{t}}{\frac{10\,000}{t} + \frac{3t}{t}}$$

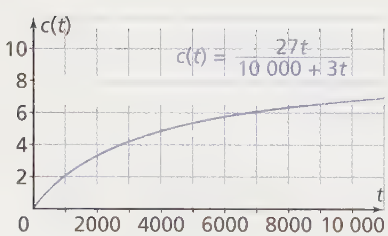
$$= \frac{27}{0 + 3}$$

$$= 9$$

The concentration will approach the concentration of the polluted water entering the pond.

Of course, this conclusion assumes that the pond has an infinite capacity, which is not a reasonable assumption!

The pond would actually overflow into the surrounding area and the water would drain away into the ground or into any nearby creeks, carrying the pollution farther afield.



Graph concentration versus time. The slopes of the tangent lines decrease over time. The curve seems to approach a limiting value.

CHECK, CONSOLIDATE, COMMUNICATE

1. Use an example to show that the derivative of a rational function is not the same as the quotient of the derivatives of its numerator and denominator.
2. Compare the quotient rule to the product rule. What is similar about the two rules? What is different?
3. Why might you need to find the first and second derivatives of a rational function? Give an example of a rational function. Then find the first and second derivatives.

KEY IDEAS

- The derivative of a quotient of two differentiable functions is not the quotient of their derivatives.
- The **quotient rule** is a rule for finding the derivative of a rational function.

  Let $h(x) = \dfrac{f(x)}{g(x)}$. If both $f'(x)$ and $g'(x)$ exist, the derivative of $h(x)$ is

  $h'(x) = \dfrac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$ where $g(x) \neq 0$.
- The quotient rule in Leibniz notation is

  $$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\left[\frac{d}{dx}f(x)\right]g(x) - \left[\frac{d}{dx}g(x)\right]f(x)}{[g(x)]^2}, \ g(x) \neq 0.$$

5.4 Exercises

A

1. Find the derivative of each rational function from first principles.

   (a) $f(x) = \dfrac{x}{3 - x}$ **(b)** $g(x) = \dfrac{x - 2}{1 + x}$ **(c)** $h(x) = \dfrac{3x^2 - 2}{x}$

2. Use the quotient rule to find $f'(x)$ for each function.

   (a) $f(x) = \dfrac{x - 3}{x + 3}$ **(b)** $f(x) = \dfrac{3x^4}{x^2 - 6}$

   (c) $f(x) = \dfrac{x^3 - 2x}{x^2 + x + 1}$ **(d)** $f(x) = \dfrac{(x + 4)(x - 5)}{2x(x + 3)}$

   (e) $f(x) = \dfrac{x^3}{1 - x}$ **(f)** $f(x) = \dfrac{ax - b}{cx - d}$

   (g) $f(x) = \dfrac{x^2 - 1}{2x + 3}$ **(h)** $f(x) = \dfrac{5x^4 + 9}{x + 2}$

   (i) $f(x) = \dfrac{1 + x^4}{x^2}$ **(j)** $f(x) = \dfrac{5 - \frac{1}{x}}{x - 3}$

3. Verify your answers for question 2 by graphing **Y1** $= f'(x)$ and **Y2** $=$ **nDeriv** $f(x)$ in the same window.

4. Find $\frac{dy}{dx}$.

   (a) $y = \frac{x + 6}{(x + 1)^2}$

   (b) $y = \frac{5x^3}{2(x - 3)}$

   (c) $y = \frac{(x + 1)(x - 4)}{(x - 2)}$

   (d) $y = \frac{x - 5}{(3x + 1)(3x - 2)}$

5. When asked to find the derivative of $f(x) = \frac{x^5 - 2x}{x^2}$, Vassili used the quotient rule. Instead of using the quotient rule, Kelly divided each term in the numerator by the denominator and then simplified. Then she found the derivative of each term. Find the derivative using each method. Explain which method you prefer, and why.

6. **Knowledge and Understanding**: Find the equation of the tangent to the graph of $f(x) = \frac{x}{3 - 2x}$ at the point where $x = 1$.

7. Find an equation for the tangent to the graph of the function at the given value of x.

   (a) $f(x) = \frac{x}{x + 3}$; $x = -5$

   (b) $f(x) = \frac{2x + 5}{5x - 1}$; $x = -1$

8. Find the point(s) where the tangent to the curve is horizontal.

   (a) $f(x) = \frac{5x}{x^2 + 1}$

   (b) $f(x) = \frac{x^2 - 2x + 4}{x^2 + 4}$

9. An object moves along a straight line. The object's position at time t is given by $s(t)$. Find the position, velocity, acceleration, and speed at the specified time.

   (a) $s(t) = \frac{2t}{t + 3}$; $t = 3$

   (b) $s(t) = t + \frac{5}{t + 2}$; $t = 1$

10. **Communication**: An object moves along a straight line so that its position, s, at t seconds is given by $s(t) = \frac{t^2 + 2t + 5}{t + 2}$. Does the object change direction at any time? Justify your answer.

11. The position, s, of an object that moves in a straight line at time t is given by $s(t) = \frac{t}{t^2 + 8}$. Determine when the object changes direction.

12. Salt water has a concentration of 10 g of salt per litre. The salt water flows into a large tank that initially holds 500 L of pure water. Twenty litres of the salt water flow into the tank per minute. Show that the concentration of salt, c, in the tank at t minutes is given by $c(t) = \frac{10t}{25 + t}$, where c is measured in grams per litre. What is the rate of change of c with respect to t?

13. **Application:** The concentration, c, of a drug in the blood t hours after the drug is taken orally is given by $c(t) = \frac{5t}{2t^2 + 7}$. When does the concentration reach its maximum value?

14. At a manufacturing plant, productivity is measured by the number of items, p, produced per employee per day over the previous 10 years. Productivity is modelled by $p(t) = \frac{25t}{t + 1}$, where t is the number of years measured from 10 years ago. Determine the rate of change of p with respect to t.

15. Find $\frac{dy}{dx}$ for $y = \frac{x^2 - 1}{2x^2 + 1}$. Determine the values of x for which $\frac{dy}{dx}$ is positive.

16. Functions u and v are differentiable functions of x, and $y = \frac{u}{v}$. Determine $\frac{dy}{dx}$ from first principles.

17. The radius of a circular juice blot on a piece of paper towel t seconds after it was first seen is modelled by $r(t) = \frac{1 + 2t}{1 + t}$, where r is measured in centimetres. Calculate

(a) the radius of the blot when it was first observed

(b) the time at which the radius of the blot was 1.5 cm

(c) the rate of increase of the area of the blot when the radius was 1.5 cm

(d) According to this model, will the radius of the blot ever reach 2 cm? Explain your answer.

18. **Check Your Understanding:** The function $P(t) = \frac{30(7t + 9)}{3t + 2}$ models the population, in thousands, of a town t years since 1985. Determine the first and second derivatives. What information do these two derivative functions give? Explain using numerical examples. Describe the population of this town.

C **19.** Find the equations of the tangents from the origin to the graph of $y = \frac{x + 8}{x + 6}$. Sketch the function and the tangent lines.

20. **Thinking, Inquiry, Problem Solving:** Choose a simple polynomial function in the form $f(x) = ax + b$. Use the quotient rule to find the derivative of the reciprocal function $\frac{1}{ax + b}$. Repeat for other polynomial functions, and devise a rule for finding the derivative of $\frac{1}{f(x)}$. Confirm your rule using first principles.

Pierre de Fermat (1601–1665)

Pierre de Fermat treated mathematics as an interesting hobby, rather than as a profession. He made contributions in calculus, number theory, and optics. Do some research on Fermat's Last Theorem. Why is it appropriate that this note on Fermat is in the margin of a math text?

ADDITIONAL ACHIEVEMENT CHART QUESTIONS

Knowledge and Understanding: Determine the derivative for $f(x) = \frac{x^2 - 1}{x^2 + 1}$. Verify your answer by graphing the derivative and using **nDeriv(** for $f(x)$.

Application: The position function of a particle moving in a straight line is $s(t) = \frac{10t^2}{32 + t^2}$, where $0 \le t \le 10$. When is the velocity a maximum?

Thinking, Inquiry, Problem Solving: The graph of $f(x) = \frac{ax + b}{(x - 1)(x - 4)}$ has a horizontal tangent line at $(2, -1)$. Find a and b. Check using a graphing calculator.

Communication: A shirt manufacturer has records that show that the unit cost, C, per shirt produced by a worker is given by $C(t) = \frac{15 + 0.6t}{5t}$, where t is the number of hours worked per day. Find approximate values for $C''(t)$ at $t = 1, 3, 5$, and 7. Describe what the numbers tell you about the cost per shirt.

The Chapter Problem

Designing a Settling Pond

Apply what you learned in this section to answer these questions about The Chapter Problem on page 342.

CP10. Determine the first and second derivatives of the first concentration function.

CP11. At what rate is the concentration changing after one hour? one day? one week? What is the average rate of change in the first week?

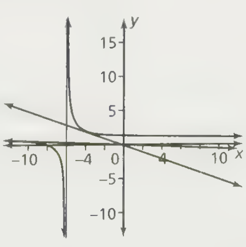
CP12. Repeat questions CP10 and CP11 for the second concentration function.

(e) all points except $x = -1$ and $x = 5$
(f) at all points
(g) all points except $x = -1$

14. Answers will vary. Example: $f(x)=\begin{cases}\dfrac{x^2+x-2}{x+2}, & x\le 3\\ x+2, & x>3\end{cases}$
15. functions in 13(b) and 13(f); satisfy all 3 conditions of continuity
16. $A = 4$
17. $\lim_{x\to a} f(x)$ exists, $f(a)$ exists, $\lim_{x\to a} f(x) = f(a)$; Examples will vary.
18. $A = B - 3$ and either $B > 1$ and $A > -2$ or $B < 1$ and $A < -2$
19. statement is true; Examples will vary.

5.4 Exercises, page 383

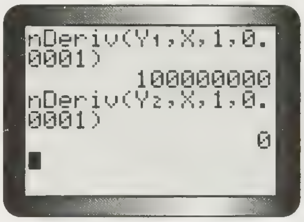
1. (a) $\frac{3}{(3-x)^2}$ (b) $\frac{3}{(1+x)^2}$ (c) $\frac{3x^2+2}{x^2}$
2. (a) $\frac{6}{(x+3)^2}$ (b) $\frac{6x^5-72x^3}{(x^2-6)^2}$
(c) $\frac{x^4+2x^3+5x^2-2}{(x^2+x+1)^2}$ (d) $\frac{2x^2+20x+30}{x^2(x+3)^2}$
(e) $\frac{3x^2-2x^3}{(1-x)^2}$ (f) $\frac{bc-ad}{(cx-d)^2}$
(g) $\frac{2x^2+6x+2}{(2x+3)^2}$ (h) $\frac{15x^4+40x^3-9}{(x+2)^2}$
(i) $\frac{2x^4-2}{x^3}$ (j) $\frac{-5x^2+2x-3}{x^2(x-3)^2}$
4. (a) $\frac{-x-11}{(x+1)^3}$ (b) $\frac{10x^3-45x^2}{2(x-3)^2}$
(c) $\frac{x^2-4x+10}{(x-2)^2}$ (d) $\frac{9x^2+90x-17}{(3x+1)^2(3x-2)^2}$
5. $\frac{3x^4+2}{x^2}$
6. $y = 3x - 2$
7. (a) $y = \frac{3}{4}x + \frac{25}{4}$ (b) $y = -\frac{3}{4}x - \frac{5}{4}$
8. (a) $\left(1, \frac{5}{2}\right)$ and $\left(-1, -\frac{5}{2}\right)$ (b) $\left(2, \frac{1}{2}\right)$ and $\left(-2, \frac{3}{2}\right)$
9. (a) position: 1, velocity: $\frac{1}{6}$, acceleration: $-\frac{1}{18}$, speed: $\frac{1}{6}$
(b) position: $\frac{8}{3}$, velocity: $\frac{4}{9}$, acceleration: $\frac{10}{27}$, speed: $\frac{4}{9}$
10. yes, when $t = 0.24$ s
11. $t = 2.83$
12. $c'(t) = \frac{250}{(25+t)^2}$
13. $t = 1.87$ h
14. $p'(t) = \frac{25}{(t+1)^2}$
15. $\frac{dy}{dx} = \frac{6x}{(2x^2+1)^2}$, positive for $x > 0$
16. $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
17. (a) 1 cm (b) $t = 1$ s (c) 0.25 cm/s
(d) no; radius will never reach 2 cm ($y = 2$ is horizontal asymptote of the graph)
18. $P'(t) = -\frac{390}{(3t+2)^2}$ is rate of change of population. $P''(t) = \frac{2340}{(3t+2)^3}$ is how the rate of change of population is changing. Examples will vary. As years pass, rate of change is increasing.
19. $y = -\frac{x}{2}$ and $y = -\frac{x}{18}$



20. Examples will vary. Rule: If $f(x) = ax + b$ and $y = \frac{1}{f(x)}$, then $y' = \frac{-a}{(ax+b)^2}$.

5.5 Exercises, page 391

1. (a) $\lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h} \neq \lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h}$
(b) corner at $x = 2$, so $\lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h} \neq \lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h}$
(c) vertical tangent at (2, 1), so slope is undefined
2. (a) **i.** all points except $x = 0$; **ii.** nowhere; **iii.** $x = 0$
(b) **i.** all points except $x = 0$ and $x = 3$; **ii.** $x = 3$; **iii.** $x = 0$
(c) **i.** all points except $x = 1$; **ii.** $x = 1$; **iii.** nowhere
3. (a) discontinuity (b) corner (c) vertical tangent
(d) cusp (e) discontinuity (f) vertical tangent
4. (a) not defined (vertical asymptote)
(b) not defined (vertical asymptote)
(c) cusp (d) corner (e) not defined (vertical asymptote)
(f) restricted domain (g) vertical asymptote
(h) discontinuity
5. (c) cusp at (2, 0) but is differentiable at $x = 1$
6. (a) all points except $x = -1$ and $x = 4$
(b) all points except $x = 2$ (c) all points except $x = 1$
7. (a) $x = 0.25$ and $x = 0$ (b) $x = -3$ and $x = 3$
(c) $1 < x < 6$
8. left and right hand limit of $\frac{f(x+h)-f(x)}{h}$ would be different
9. (a) not defined at $x = 1$ (vertical asymptote)
(b)



(c) $f(x)$: 100 000 000; $g(x)$: 0 (d) Answers will vary.
10. no
12. discontinuity or restricted domain, graph has corner or cusp, graph has vertical tangent; Examples will vary.
13. (a) $a = 4 + 2b$ (b) $a = -8, b = -6$
14. (a) $6b = 2a + 5$ (b) $a = \frac{11}{4}, b = \frac{7}{4}$
15. converse is not true. Counterexamples will vary.

Exercise 5.6, page 401

1. 1.5, min.
2. 6 cm × 6 cm
3. 3.16 m × 6.33 m
4. 8 m × 12 m; 48 m
5. 17.3 m × 34.6 m
6. 25 m × 40 m
7. 60 m × 80 m
8. 50 m × 75 m
9. 4 m × 8 m
10. 20 km/h
11. 11.3 cm × 17.0 cm
12. 8.5 cm × 14.1 cm
13. 5 cm × 10 cm × 10 cm
14. 2.38 m × 2.38 m × 2.65 m
15. 14.94 cm × 14.94 cm × 22.40 cm
16. 0.585 m × 1.170 m × 0.438 m
17. radius 2.75 m, height 21.05 m
18. radius 0.66 m, height 1.54 m
19. (a) 0.60 m × 1.29 m × 1.32 m
(b) 0.50 m × 1.41 m × 1.41 m