5.5 Differentiability of Rational and Other Functions

SETTING THE STAGE

Explore the concepts in this lesson in more detail using Exploration 10 on page 578.

A function f is differentiable at $x = a$ if $f'(a)$ exists. A polynomial function is differentiable at every number in the domain. Consider two questions.

- Are there functions for which the derivative does not exist at one or more numbers in the domain?
- If so, what properties of these functions cause the derivative not to exist at these numbers?

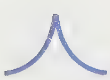
In this section, you will examine the differentiability of rational functions as well as other types of functions.

EXAMINING THE CONCEPT

Differentiating a Rational Function

If $f'(a)$ exists, then $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ exists. Both the one-sided limits exist and are equal: $\lim\limits_{h \to 0^-} \dfrac{f(a+h) - f(a)}{h} = \lim\limits_{h \to 0^+} \dfrac{f(a+h) - f(a)}{h}$. Also, you can draw a tangent line at $x = a$ if $f'(a)$ exists.

corner



cusp

The Differentiability of a Function

A function is not differentiable at $x = a$ when

1. the graph of the function has a discontinuity at a
2. the graph of the function has a corner or a cusp
3. the line $x = a$ is a vertical tangent

In the next three examples, you will examine each case in more detail.

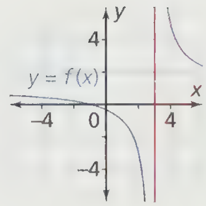
Example 1 **Functions Whose Derivatives Do Not Exist Because of a Discontinuity or a Restricted Domain**

Explain why each function is not differentiable at $x = 3$.

(a) $f(x) = \dfrac{x+1}{x-3}$ **(b)** $g(x) = \dfrac{x^2 - 9}{x - 3}$

(c) $h(x) = \sqrt{3 - x}$ **(d)** $m(x) = \begin{cases} 3x - 1 & \text{if } x \le 3 \\ 1 & \text{if } x > 3 \end{cases}$

Solution

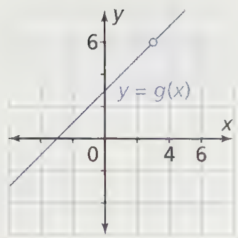


(a) $f(x) = \dfrac{x+1}{x-3}$

This function is not defined at $x = 3$.

The line $x = 3$ is a vertical asymptote. The function is discontinuous at $x = 3$.

You cannot draw a tangent line at $x = 3$. $f'(3)$ does not exist.

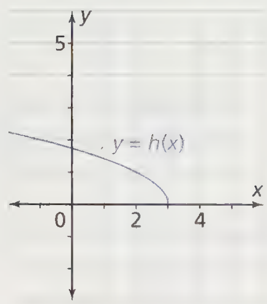


(b) $g(x) = \dfrac{x^2 - 9}{x - 3}$

This function is not defined at $x = 3$.

The graph has a hole at $x = 3$. The function is discontinuous at $x = 3$.

You cannot draw a tangent line at $x = 3$. $g'(x)$ does not exist.



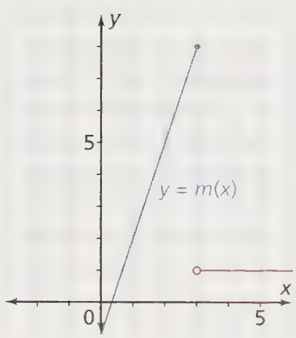
(c) $h(x) = \sqrt{3 - x}$

This function is not defined for $x > 3$, so $f(3 + h)$ is not defined.

$$\lim_{h \to 0^+} \frac{f(3+h) - f(3)}{h} \text{ does not exist.}$$

$$\lim_{h \to 0^-} \frac{f(3+h) - f(3)}{h} = -\infty.$$

Since these limits are not equal, $h'(3)$ does not exist.



(d) $m(x) = \begin{cases} 3x - 1 & \text{if } x \le 3 \\ 1 & \text{if } x > 3 \end{cases}$

The one-sided limits as $x \to 3$ are not the same. That is,

$\lim_{x \to 3^-} m(x) \neq \lim_{x \to 3^+} m(x)$. The function is discontinuous at $x = 3$.

$m'(3)$ does not exist.

Example 2 **A Function Whose Derivative Does Not Exist at a Corner in Its Graph**

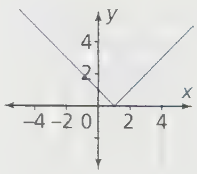
Let $f(x) = |x - 1|$. Is this function differentiable at $x = 1$?

x	$f(x)$
–3	4
–2	3
–1	2
0	1
1	0
2	1
3	2
4	3

Solution

Create a table for the function. Then graph the function.

The graph has a corner at (1, 0).



$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = -1$$

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = 1$$

Since these limits are not the same, $f'(1)$ does not exist.

The slope changes abruptly from -1 to 1 at (1, 0). You cannot draw a unique tangent line at this point.

The function is not differentiable at $x = 1$.

An extreme case of a corner is a **cusp**. A cusp occurs where the slopes of the secant lines approach ∞ from one side and $-\infty$ from the other side.

Example 3

A Function Whose Derivative Does Not Exist at a Cusp in Its Graph

Where is $f(x) = (3x - 6)^{\frac{2}{3}}$ differentiable?

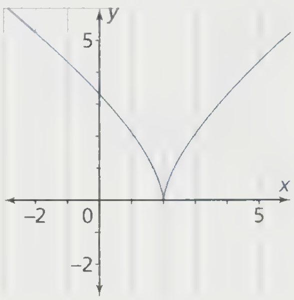
x	$f(x)$
–3	6.0822
–2	5.2415
–1	4.3267
0	3.3019
1	2.0801
2	0.0000
3	2.0801
4	3.3019
5	4.3267

Solution

Make a table. (The values for the function in the table are rounded to four decimal places.) Then graph the function.

$$\lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h} = -\infty, \text{ but}$$

$$\lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} = \infty$$



So, $f'(2)$ does not exist.

The graph has a cusp at (2, 0).

The function is not differentiable at $x = 2$.

Example 4

A Function with a Vertical Tangent

Show that $f(x) = \sqrt[3]{x - 1}$ is not differentiable at $x = 1$.

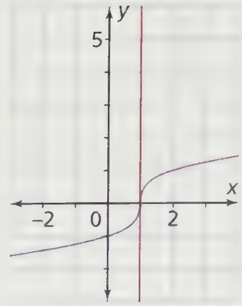
Solution

Make a table. (The values for the function in the table are rounded to four decimal places.) Then graph the function.

x	$f(x)$
–4	–1.7100
–3	–1.5874
–2	–1.4422
–1	–1.2599
0	–1.0000
1	0.0000
2	1.0000
3	1.2599
4	1.4422

A tangent line can be drawn at point (1, 0).

However, this tangent is vertical, so its slope is undefined.
The slope does not exist, and $f'(1)$ does not exist.
The function is not differentiable at $x = 1$.



Maximum and minimum values can occur at corners and cusps. At these points, the derivative does not exist. To account for all critical points, we must also consider the points where the derivative does not exist. Points of inflection can occur at corners. Also, vertical tangents occur at points of inflection. To account for all points of inflection, we must also consider where the second derivative does not exist.

Critical Points and Points of Inflection

For any function $f(x)$, at a critical number $f'(c) = 0$ or $f'(c)$ does not exist. At a point of inflection, $f''(c) = 0$ or $f''(c)$ does not exist.

You may have noticed that the descriptions of differentiability and continuity are similar. In fact, they are closely connected. Every differentiable function is also continuous.

Continuity and Differentiability

The function $f(x)$ is continuous at $x = a$ if $f(x)$ is differentiable at $x = a$.

Example 5 Showing That a Differentiable Function Is Also Continuous

Show that $f(x)$ is continuous at $x = a$ if $f(x)$ is differentiable at $x = a$.

Solution

Show that $\lim\limits_{h \to 0} f(a + h) = f(a)$, which means $\lim\limits_{h \to 0} [f(a + h) - f(a)] = 0$.

For $h \neq 0$ and $(a + h)$ in the domain of the function,

$$\begin{aligned}\lim_{h \to 0} [f(a + h) - f(a)] &= \lim_{h \to 0} \left[\frac{f(a + h) - f(a)}{h} \times h\right] \\ &= \left[\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}\right] \times (\lim_{h \to 0} h) \\ &= f'(a) \times 0 \\ &= 0\end{aligned}$$

So f is continuous at $x = a$ if $f'(a)$ exists.

CHECK, CONSOLIDATE, COMMUNICATE

1. Use examples to show how a rational function can fail to be differentiable at one or more numbers x.
2. Give an example of a function that has a corner in its graph. Explain why the function is not differentiable at the corner point. Repeat for a function with a cusp in its graph.
3. Explain why a function with a vertical tangent $x = a$ is not differentiable at a. Use an example in your explanation.
4. How are differentiability and continuity related?

KEY IDEAS

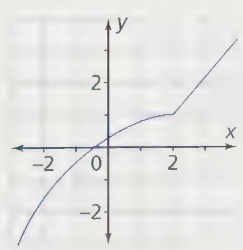
- A function $f(x)$ is differentiable at a number a in its domain if $f'(a)$ exists.
- A function will not be differentiable at a when
  - the graph of the function has a discontinuity at a
  - the graph of the function has a corner or a cusp at a
  - the line $x = a$ is a vertical tangent
- A cusp occurs where the slopes of the secant lines approach ∞ from one side and $-\infty$ from the other side.
- A function is continuous at number a in the domain of the function if the function is differentiable at $x = a$.
- For any function $f(x)$, at a critical number, $f'(c) = 0$ or $f'(c)$ does not exist.
- For any function $f(x)$, at a point of inflection, $f''(c) = 0$ or $f''(c)$ does not exist.

5.5 Exercises

A

1. Explain why each function is not differentiable at $x = 2$.

(a)



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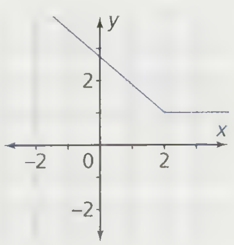

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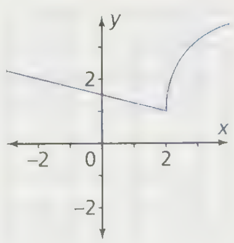
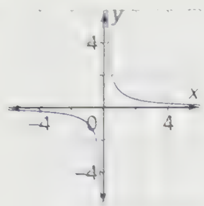


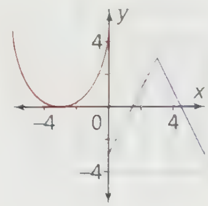
2. Where is each function

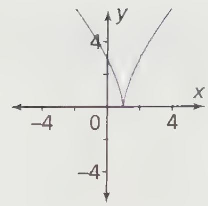
   i. differentiable?

   ii. continuous but not differentiable?

   iii. neither continuous nor differentiable?

(a) 

(b) 

(c) 

3. Each function is *not* differentiable at $x = 1$. Determine whether the reason is a discontinuity, a corner, a cusp, or a vertical tangent.

   (a) $f(x) = \dfrac{x^2 - 1}{x - 1}$

   (b) $f(x) = \frac{1}{2}|1 - x| + 1$

   (c) $f(x) = \dfrac{x + 4}{x - 1}$

   (d) $f(x) = (x - 1)^{\frac{2}{3}}$

   (e) $f(x) = 2\sqrt{1 - x}$

   (f) $f(x) = (x - 1)^{\frac{1}{3}}$

B

4. Explain why each function is not differentiable at the given x-value(s).

   (a) $f(x) = \dfrac{1}{(3 - x)^2}$ at $x = 3$

   (b) $f(x) = \dfrac{x - 4}{16 - x^2}$ at $x = \pm 4$

   (c) $f(x) = \sqrt[3]{(2x - 5)^2}$ at $x = 2.5$

   (d) $f(x) = \frac{2}{3}|4 - 5x| - 1$ at $x = 0.8$

   (e) $f(x) = \dfrac{2x - 1}{4x - 9}$ at $x = 2.25$

   (f) $f(x) = 5\sqrt{3 - 2x}$ at $x = 1.5$

   (g) $f(x) = \sqrt[3]{5 - 4x}$ at $x = 1.25$

   (h) $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x \le 2 \\ 6 - x & \text{if } x > 2 \end{cases}$

5. **Knowledge and Understanding**: Determine which functions are differentiable at $x = 1$. Give reasons for your choices.

   (a) $f(x) = \dfrac{3x}{1 - x^2}$

   (b) $g(x) = \dfrac{x - 1}{x^2 + 5x - 6}$

   (c) $h(x) = \sqrt[3]{(x - 2)^2}$

   (d) $m(x) = |3x - 3| - 1$

6. Find all x-values for which each function is differentiable. Explain your answers.

(a) $f(x) = \dfrac{x^3 + 1}{x^2 - 3x - 4}$

(b) $f(x) = 1 - 4|2 - x|$

(c) $f(x) = \dfrac{2 - 3x}{x - 1}$

7. At what x-values is each function *not* differentiable? Explain.

(a) $f(x) = \dfrac{3}{4x^2 - x}$ (b) $f(x) = \dfrac{x^2 - x - 6}{x^2 - 9}$ (c) $f(x) = \sqrt{x^2 - 7x + 6}$

8. **Communication:** Explain, with an example, why a function whose graph has a cusp is not differentiable at the point where the cusp occurs.

9. (a) Explain why $f(x) = \dfrac{1}{x - 1}$ and $g(x) = \dfrac{1}{(x - 1)^2}$ are not differentiable at $x = 1$.

(b) Use a graphing calculator to obtain nDeriv($f(x)$, x, 1, 0.0001) and nDeriv($g(x)$, x, 1, 0.0001).

(c) Explain your results for (b) by calculating $\dfrac{f(a + h) - f(a - h)}{2h}$ and $\dfrac{g(a + h) - g(a - h)}{2h}$ for $a = 1$ and $h = 0.0001$.

(d) Find the derivatives of other functions at x-values where the functions are not differentiable using the **nDeriv(** operation. Explain each erroneous result using the function definition

$$\text{nDeriv}(f(x), x, a, h) = \frac{f(a + h) - f(a - h)}{2h}.$$

10. **Application:** Determine whether the derivative of $f(x) = \begin{cases} \dfrac{1}{x} & \text{if } x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$ exists at $x = 1$.

11. Use the limit definition of the derivative to show that $f'(3)$ does not exist for $f(x) = |x - 3|$.

12. **Check Your Understanding:** Explain, with examples, the three ways in which a function may not be differentiable at $x = a$.

13. Let $f(x) = \begin{cases} 2x + b & \text{if } x < 1 \\ \dfrac{a}{x + 1} & \text{if } x \geq 1 \end{cases}$

(a) Find a relation between a and b if the function is continuous for all values of x.

(b) Find the values of a and b that make the function both continuous and differentiable for all values of x.

14. Repeat question 13 for $f(x) = \begin{cases} 5 - x & \text{if } x < 2 \\ \dfrac{ax + 2}{bx - 1} & \text{if } x \geq 2 \end{cases}$

15. **Thinking, Inquiry, Problem Solving:** You have seen that a function that is differentiable is continuous. Investigate whether the converse is true: is a function differentiable at $x = a$ if the function is continuous at $x = a$? Give examples to support your arguments.

ADDITIONAL ACHIEVEMENT CHART QUESTIONS

Knowledge and Understanding: Find all values of x for which the function is differentiable.

(a) $y = \dfrac{x^3 - 8}{x^2 - 4x - 5}$

(b) $h(x) = \sqrt[3]{3x - 6} + 4$

(c) $y = x|x|$

(d) $g(x) = \begin{cases} (x + 1) & \text{if } x \leq 0 \\ 2x + 1 & \text{if } 0 < x < 3 \\ (4 - x)^2 & \text{if } x \geq 3 \end{cases}$

Application: Use the limit definition of the derivative to show that $f'(5)$ does not exist for $f(x) = |x - 5|$.

Thinking, Inquiry, Problem Solving: Let $f(x) = \begin{cases} 3x^2 & \text{if } x \leq 1 \\ ax + b & \text{if } x > 1 \end{cases}$

where a and b are constants.

(a) What is the relation between a and b if the function is continuous for all x?

(b) Find the unique values for a and b that will make the function both continuous and differentiable.

Communication: Explain why $f(x) = \begin{cases} 0 & \text{if } -1 \leq x < 0 \\ 2 & \text{if } 0 \leq x \leq 1 \end{cases}$

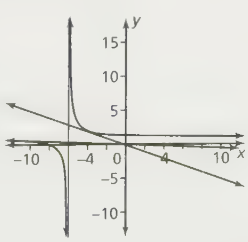
is not the derivative of any function on the interval $-1 \leq x \leq 1$.

(e) all points except $x = -1$ and $x = 5$
(f) at all points
(g) all points except $x = -1$

14. Answers will vary. Example: $f(x)=\begin{cases}\dfrac{x^2+x-2}{x+2}, & x \le 3\\ x+2, & x > 3\end{cases}$
15. functions in 13(b) and 13(f); satisfy all 3 conditions of continuity
16. $A = 4$
17. $\lim_{x \to a} f(x)$ exists, $f(a)$ exists, $\lim_{x \to a} f(x) = f(a)$; Examples will vary.
18. $A = B - 3$ and either $B > 1$ and $A > -2$ or $B < 1$ and $A < -2$
19. statement is true; Examples will vary.

5.4 Exercises, page 383

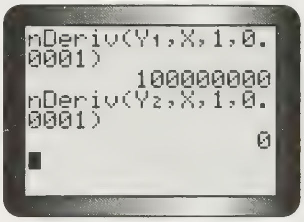
1. (a) $\frac{3}{(3-x)^2}$ (b) $\frac{3}{(1+x)^2}$ (c) $\frac{3x^2+2}{x^2}$
2. (a) $\frac{6}{(x+3)^2}$ (b) $\frac{6x^5-72x^3}{(x^2-6)^2}$
(c) $\frac{x^4+2x^3+5x^2-2}{(x^2+x+1)^2}$ (d) $\frac{2x^2+20x+30}{x^2(x+3)^2}$
(e) $\frac{3x^2-2x^3}{(1-x)^2}$ (f) $\frac{bc-ad}{(cx-d)^2}$
(g) $\frac{2x^2+6x+2}{(2x+3)^2}$ (h) $\frac{15x^4+40x^3-9}{(x+2)^2}$
(i) $\frac{2x^4-2}{x^3}$ (j) $\frac{-5x^2+2x-3}{x^2(x-3)^2}$
4. (a) $\frac{-x-11}{(x+1)^3}$ (b) $\frac{10x^3-45x^2}{2(x-3)^2}$
(c) $\frac{x^2-4x+10}{(x-2)^2}$ (d) $\frac{9x^2+90x-17}{(3x+1)^2(3x-2)^2}$
5. $\frac{3x^4+2}{x^2}$
6. $y = 3x - 2$
7. (a) $y = \frac{3}{4}x + \frac{25}{4}$ (b) $y = -\frac{3}{4}x - \frac{5}{4}$
8. (a) $\left(1, \frac{5}{2}\right)$ and $\left(-1, -\frac{5}{2}\right)$ (b) $\left(2, \frac{1}{2}\right)$ and $\left(-2, \frac{3}{2}\right)$
9. (a) position: 1, velocity: $\frac{1}{6}$, acceleration: $-\frac{1}{18}$, speed: $\frac{1}{6}$
(b) position: $\frac{8}{3}$, velocity: $\frac{4}{9}$, acceleration: $\frac{10}{27}$, speed: $\frac{4}{9}$
10. yes, when $t = 0.24$ s
11. $t = 2.83$
12. $c'(t) = \frac{250}{(25+t)^2}$
13. $t = 1.87$ h
14. $p'(t) = \frac{25}{(t+1)^2}$
15. $\frac{dy}{dx} = \frac{6x}{(2x^2+1)^2}$, positive for $x > 0$
16. $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
17. (a) 1 cm (b) $t = 1$ s (c) 0.25 cm/s
(d) no; radius will never reach 2 cm ($y = 2$ is horizontal asymptote of the graph)
18. $P'(t) = -\frac{390}{(3t+2)^2}$ is rate of change of population. $P''(t) = \frac{2340}{(3t+2)^3}$ is how the rate of change of population is changing. Examples will vary. As years pass, rate of change is increasing.
19. $y = -\frac{x}{2}$ and $y = -\frac{x}{18}$



20. Examples will vary. Rule: If $f(x) = ax + b$ and $y = \frac{1}{f(x)}$, then $y' = \frac{-a}{(ax+b)^2}$.

5.5 Exercises, page 391

1. (a) $\lim_{h \to 0^-} \frac{f(2+h)-f(2)}{h} \neq \lim_{h \to 0^+} \frac{f(2+h)-f(2)}{h}$
(b) corner at $x = 2$, so $\lim_{h \to 0^-} \frac{f(2+h)-f(2)}{h} \neq \lim_{h \to 0^+} \frac{f(2+h)-f(2)}{h}$
(c) vertical tangent at (2, 1), so slope is undefined
2. (a) **i.** all points except $x = 0$; **ii.** nowhere; **iii.** $x = 0$
(b) **i.** all points except $x = 0$ and $x = 3$; **ii.** $x = 3$; **iii.** $x = 0$
(c) **i.** all points except $x = 1$; **ii.** $x = 1$; **iii.** nowhere
3. (a) discontinuity (b) corner (c) vertical tangent
(d) cusp (e) discontinuity (f) vertical tangent
4. (a) not defined (vertical asymptote)
(b) not defined (vertical asymptote)
(c) cusp (d) corner (e) not defined (vertical asymptote)
(f) restricted domain (g) vertical asymptote
(h) discontinuity
5. (c) cusp at (2, 0) but is differentiable at $x = 1$
6. (a) all points except $x = -1$ and $x = 4$
(b) all points except $x = 2$ (c) all points except $x = 1$
7. (a) $x = 0.25$ and $x = 0$ (b) $x = -3$ and $x = 3$
(c) $1 < x < 6$
8. left and right hand limit of $\frac{f(x+h)-f(x)}{h}$ would be different
9. (a) not defined at $x = 1$ (vertical asymptote)
(b)



(c) $f(x)$: 100 000 000; $g(x)$: 0 (d) Answers will vary.
10. no
12. discontinuity or restricted domain, graph has corner or cusp, graph has vertical tangent; Examples will vary.
13. (a) $a = 4 + 2b$ (b) $a = -8$, $b = -6$
14. (a) $6b = 2a + 5$ (b) $a = \frac{11}{4}$, $b = \frac{7}{4}$
15. converse is not true. Counterexamples will vary.

Exercise 5.6, page 401

1. 1.5, min.
2. 6 cm × 6 cm
3. 3.16 m × 6.33 m
4. 8 m × 12 m; 48 m
5. 17.3 m × 34.6 m
6. 25 m × 40 m
7. 60 m × 80 m
8. 50 m × 75 m
9. 4 m × 8 m
10. 20 km/h
11. 11.3 cm × 17.0 cm
12. 8.5 cm × 14.1 cm
13. 5 cm × 10 cm × 10 cm
14. 2.38 m × 2.38 m × 2.65 m
15. 14.94 cm × 14.94 cm × 22.40 cm
16. 0.585 m × 1.170 m × 0.438 m
17. radius 2.75 m, height 21.05 m
18. radius 0.66 m, height 1.54 m
19. (a) 0.60 m × 1.29 m × 1.32 m
(b) 0.50 m × 1.41 m × 1.41 m