6.3 Differentiation Techniques: Combining the Differentiation Rules

SETTING THE STAGE

In the previous section, you used the chain rule to differentiate. In this section, you will extend this rule to more complex functions. You will need to apply many of the differentiation rules in different combinations. Sometimes decomposing a complex function into simpler functions can help you to differentiate the function.

EXAMINING THE CONCEPT

Combining Differentiation Rules

At this point, you know how to differentiate many simple functions by applying these rules:

the constant rule	$\frac{d}{dx}(c) = 0$
the constant multiple rule	$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$
the power rule	$\frac{d}{dx}(x^n) = nx^{n-1}$
the sum and difference rules	$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$
the product rule	$\frac{d}{dx}[f(x) \cdot g(x)] = \frac{d}{dx}[f(x)] \cdot g(x) + \frac{d}{dx}[g(x)] \cdot f(x)$
the quotient rule	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}[f(x)] \cdot g(x) - \frac{d}{dx}[g(x)] \cdot f(x)}{[(g(x)]^2}$
the chain rule	$\frac{d}{dx}[f(g(x))] = \frac{d[f(g(x))]}{d[g(x)]} \cdot \frac{d[g(x)]}{dx}$
the chain rule with power rule	$\frac{d}{dx}[g(x)^n] = n[g(x)]^{n-1} \cdot \frac{d}{dx}[g(x)]$, where n is a constant

Differentiating a complex function often means rewriting it in terms of the sum, difference, product, quotient, or power of polynomials.

The first step of differentiating $r = \sqrt{1 + (c - 3)^2}$ in Example 3 in the last section was to simplify the expression under the radical. Simplifying first is not always the best approach. Sometimes it is better to decompose a function into simpler functions.

Example 1 Differentiating Using Decomposition

Use decomposition to determine $\frac{dr}{dc}$.

Solution

The sequence of operations helps determine the order in which to differentiate. The last operation to be performed is the first operation to be dealt with when differentiating. First express the radical as a power.

$$r = \sqrt{1 + (c - 3)^2} = [1 + (c - 3)^2]^{\frac{1}{2}}$$

r can be defined using three different functions: $f(c) = c - 3$, $g(v) = 1 + v^2$, and $h(u) = u^{\frac{1}{2}}$, where $v = c - 3$, $u = 1 + v^2$, and $r = u^{\frac{1}{2}}$.

Therefore, $r = h(u) = h(g(v)) = h(g(f(c)))$.

$$f'(c) = 1 \quad g'(v) = 2v \quad h'(u) = \frac{1}{2}u^{-\frac{1}{2}}$$

Differentiate each component function with respect to its independent variable.

$$\frac{dr}{dc} = h'(g(f(c))) \cdot g'(f(c)) \cdot f'(c)$$

Apply the chain rule to find $\frac{dr}{dc}$.

$$= \frac{1}{2}[g(f(c))]^{-\frac{1}{2}} \cdot 2f(c) \cdot 1$$

$$= \frac{1}{2}[g(c - 3)]^{-\frac{1}{2}} \cdot 2(c - 3)$$

Substitute the expression for each derivative.

$$= \frac{1}{2}[1 + (c - 3)]^{-\frac{1}{2}} \cdot 2(c - 3)$$

Simplify.

$$= \frac{c - 3}{\sqrt{1 + (c - 3)^2}}$$

As you become more comfortable with the chain rule, you can simplify how you find the derivative. Here is a general rule for finding the derivative of a composite function $f(g(x))$, where the inner function g and the outer function f are differentiable.

$$\frac{d}{dx}[f(g(x))] = \underbrace{f'(g(x))} \cdot \underbrace{g'(x)}$$

derivative of the outside evaluated at the inside

derivative of the inside

The next four examples show some techniques for determining the derivatives of functions involving products, quotients, and composites.

Example 2 The Derivative of a Complex Product

Determine $f'(x)$, where $f(x) = (x^2 + 3)^4(4x - 5)^3$.

Solution

Apply the product rule and the chain rule.

$$f'(x) = \frac{d}{dx}[(x^2 + 3)^4] \cdot (4x - 5)^3 + \frac{d}{dx}[(4x - 5)^3] \cdot (x^2 + 3)^4$$

$$= [4(x^2 + 3)^3(2x)] \cdot (4x - 3)^3 + [3(4x - 5)^2(4)] \cdot (x^2 + 3)^4 \quad \text{Simplify.}$$

$$= 8x(x^2 + 3)^3 \cdot (4x - 5)^3 + 12(4x - 5)^2 \cdot (x^2 + 3)^4 \quad \text{Factor.}$$

$$= 4(x^2 + 3)^3(4x - 5)^2[2x(4x - 5) + 3(x^2 + 3)] \quad \text{Simplify.}$$

$$= 4(x^2 + 3)^3(4x - 5)^2(11x^2 - 10x + 9)$$

Example 3 The Derivative of a Complex Quotient

Determine $g'(x)$, where $g(x) = \dfrac{2x}{\sqrt[3]{x^2+4}}$.

Solution

$$g(x) = \frac{2x}{\sqrt[3]{x^2+4}} = \frac{2x}{(x^2+4)^{\frac{1}{3}}}$$

Express $g(x)$ with a rational exponent.

$$g(x) = \frac{\frac{d}{dx}[2x]\cdot(x^2+4)^{\frac{1}{3}} - \frac{d}{dx}\left[(x^2+4)^{\frac{1}{3}}\right]\cdot(2x)}{\left[(x^2+4)^{\frac{1}{3}}\right]^2}$$

Apply the quotient rule and the chain rule.

$$= \frac{(2)(x^2+4)^{\frac{1}{3}} - \left[\frac{1}{3}(x^2+4)^{-\frac{2}{3}}(2x)\right](2x)}{(x^2+4)^{\frac{2}{3}}}$$

Factor the numerator.

$$= \frac{1}{3}(x^2+4)^{-\frac{2}{3}}\left[\frac{6(x^2+4)-(4x^2)}{(x^2+4)^{\frac{2}{3}}}\right]$$

Simplify.

$$= \frac{2x^2+24}{3(x^2+4)^{\frac{4}{3}}}$$

Another way to differentiate a quotient is to express it as a product using a negative exponent. In this case, $g(x) = (2x)(x^2+4)^{-\frac{1}{3}}$. Often this makes it easier to find and simplify the derivative. Try to write expressions in terms of the sum, difference, product, and powers of polynomials.

Example 4 The Derivative of a Complex Power

Determine the derivative of $g(x) = \left(\dfrac{1+x^2}{1-x^2}\right)^{10}$.

Solution

There are several approaches to this problem. You could decompose the function and express it as $g(x) = \dfrac{(1+x^2)^{10}}{(1-x^2)^{10}}$, and then apply the quotient rule and the chain rule. Or you could express the function as the product $g(x) = (1+x^2)^{10}(1-x^2)^{-10}$, and then apply the product and the chain rules. In this case we will use the chain rule with the power rule, where $\dfrac{1+x^2}{1-x^2}$ is the inner function.

$$g'(x) = \frac{d\left[\left(\frac{1+x^2}{1-x^2}\right)^{10}\right]}{d\left(\frac{1+x^2}{1-x^2}\right)} \cdot \frac{d}{dx}\left(\frac{1+x^2}{1-x^2}\right)$$ Apply the chain rule and the quotient rule

$$g'(x) = 10\left(\frac{1+x^2}{1-x^2}\right)^9 \frac{d}{dx}\left(\frac{1+x^2}{1-x^2}\right)$$

$$= 10\left(\frac{1+x^2}{1-x^2}\right)^9\left[\frac{2x(1-x^2)-(-2x)(1+x^2)}{(1-x^2)^2}\right]$$ Expand.

$$= 10\left(\frac{1+x^2}{1-x^2}\right)^9\left[\frac{2x-2x^3+2x+2x^3}{(1-x^2)^2}\right]$$ Simplify.

$$= 10\left(\frac{1+x^2}{1-x^2}\right)^9\left[\frac{4x}{(1-x^2)^2}\right]$$

$$= \frac{40x(1+x^2)^9}{(1-x^2)^{11}}$$

Example 5 **Differentiating a Complex Function**

Determine the derivative of $f(t) = \left(\frac{\sqrt[3]{5+3t}}{1-t^2}\right)^2$.

Solution

Express $f(t)$ as a product of simpler functions. Express the radical as a power and the quotient as a product, and then apply the exponent law for power of a power.

$$f(t) = \left[(5+3t)^{\frac{1}{3}}(1-t^2)^{-1}\right]^2$$ Express the radical as a power and express the quotient as a product.

$$= (5+3t)^{\frac{2}{3}}(1-t^2)^{-2}$$ Simplify.

When you differentiate a complex function, apply the differentiation rules in the order that is opposite to the order of operations. In this example, the last operation is the multiplication of two expressions, so apply the product rule first.

$$f(t) = (5+3t)^{\frac{2}{3}}(1-t^2)^{-2}$$ Apply the product rule and then the chain rule.

$$f'(t) = \frac{d}{dt}\left[(5+3t)^{\frac{2}{3}}\right](1-t^2)^{-2} + \frac{d}{dt}[(1-t^2)^{-2}](5+3t)^{\frac{2}{3}}$$

$$= \frac{2}{3}(5+3t)^{-\frac{1}{3}}(3)(1-t^2)^{-2} + (-2)(1-t^2)^{-3}(-2t)(5+3t)^{\frac{2}{3}}$$

$$= 2(5+3t)^{-\frac{1}{3}}(1-t^2)^{-2} + 4t(1-t^2)^{-3}(5+3t)^{\frac{2}{3}}$$ Simplify and factor.

$$= 2(5+3t)^{-\frac{1}{3}}(1-t^2)^{-3}[(1-t^2) + 2t(5+3t)]$$

$$= 2(5+3t)^{-\frac{1}{3}}(1-t^2)^{-3}(1-t^2+10t+6t^2)$$ Rewrite using positive exponents.

$$= \frac{2(5t^2+10t+1)}{(5+3t)^{\frac{1}{3}}(1-t^2)^3}$$

This method avoids introducing new variables, but it may lead to more minor errors. You should choose the approach that works best for you.

TI-83 Plus Help: See the Technology Appendix on page 595 for details about this function.

You can check the reasonableness of a solution by comparing the graph of $f'(x)$ with the graph of $y = f(x)$ that you could draw using the TI-83 Plus and the numerical derivative function, **nDeriv(**.

CHECK, CONSOLIDATE, COMMUNICATE

1. Which differentiation rules would you use and in which order would you use them to find $f'(t)$ if $f(t) = \frac{(\sqrt{2t+3}+5)^3 + 2t}{1+t^2}$? Explain.
2. Why would you *not* use the chain rule with the power rule to differentiate $y = 2^x$?
3. What is another rule or rules that you could use instead of the quotient rule for differentiating most complex functions?

KEY IDEAS

- When differentiating a complex expression, rewrite functions as combinations of simpler functions and, if possible,
  - express each radical as a power
  - apply the exponent law of a power of a power
  - express each product (or quotient) as the product of two expressions
- When you differentiate a complex function, apply the differentiation rules in the order that is opposite to the order of operations.
- There are often several approaches for differentiating complex functions. Choose the method that works best for you.

6.3 Exercises

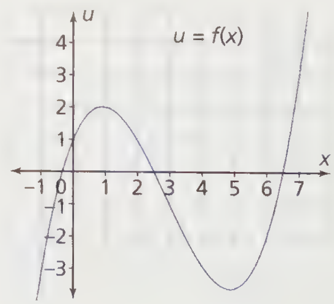
A

1. Given $u = s^2 - 1$, $y = \frac{2}{u}$, and $s = 3 - x$, determine each derivative.

B

(a) $\frac{du}{ds}$ (b) $\frac{dy}{du}$ (c) $\frac{ds}{dx}$ (d) $\frac{dy}{dx}$, evaluate at $x = 1$

2. **Communication:** Copy the graph of $u = f(x)$. On the same set of axes, graph the derivative of $g(x) = [f(x)]^2 - 4$ without sketching g first. Justify your graph.



3. Given $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2 - 3x}$,
   (a) express g as the composition of f and another function
   (b) use this composition to determine $g'(x)$

4. Given $f(x) = \frac{(2x-3)^2 + 5}{2x - 3}$,
   (a) express f as the composition of two simpler functions
   (b) use this composition to determine $f'(x)$

5. Given $g(x) = \sqrt{2x - 3} + 5(2x - 3)$,
   (a) express f as the composition of two simpler functions
   (b) use this composition to determine $g'(x)$

6. Determine the derivative of each function.
   (a) $f(x) = (2x - 5)^3(3x^2 + 4)^5$
   (b) $g(x) = (8x^3)(4x^2 + 2x - 3)^5$
   (c) $y = (5 + x)^2(4 - 7x^3)^6$
   (d) $h(x) = \frac{6x - 1}{(3x + 5)^4}$
   (e) $y = \frac{(2x^2 - 5)^3}{(x + 8)^2}$
   (f) $f(x) = \frac{-3x^4}{\sqrt{4x - 8}}$
   (g) $g(x) = \left(\frac{2x + 5}{6 - x^2}\right)^4$
   (h) $y = \left[\frac{1}{(4x + x^2)^3}\right]^3$
   (i) $h(x) = \frac{-4\sqrt{3x + 2}}{(2x - x^3)^2}$
   (j) $y = \sqrt{\frac{x^2 + 1}{x^2 - 1}}$
   (k) $f(x) = [2x + (3x^2 - 5x)^3]^5$
   (l) $g(x) = \sqrt{2x + \sqrt{x^3}}$

7. (a) Let $y = \frac{\sqrt{x}}{\sqrt{x} + 1}$. Determine $\frac{dy}{dx}$. (b) Let $y = \frac{\sqrt{2x + 1}}{\sqrt{x + 3}}$. Determine $\frac{dy}{dx}$.

8. For $y = 3\{x - [x - 3(x + 2)^2]^{-1}\}$, determine $\frac{dy}{dx}$.

9. **Knowledge and Understanding:** Let $s = 3 - 2t - [t^{-2} - (3t + 5)^4]^5$. Determine $\frac{ds}{dt}$.

10. **Application:** Consider $y = f(x)$ and $g(x) = \sqrt{f(x)}$. Given $f'(a) = 0$, $f(a) > 0$, and $f''(a) > 0$, prove that g has a relative minimum at $x = a$.

11. Given $f(t) = \left(\frac{\sqrt[3]{1 - 2t}}{1 + t^2}\right)^2$, determine $f'(0)$.

12. Let $y = -2(x + [2x - 5(x - 2)^3]^{-1})$. Determine $\frac{dy}{dx}$.

13. Given $f(x) = 3x - 1$ and $g(x) = \sqrt{[f(x)]^2 - 1}$, determine $g'(x)$.

14. Find the equation of the tangent line to the curve $y = 4x^2(3x^2 - 5x)^3$ at point (2, 128).

15. Given $f(x) = 5 - \frac{1}{\sqrt{x^2 - 1}}$, determine the intervals on which f is decreasing.

16. Let $f(x) = \left(\sqrt{x - 2}\right)(2x + 5)^{-1}$. Determine the extreme points on its graph.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-1	1	18	-5	-15
0	-2	5	-1	-11
1	-1	-4	3	-7
2	4	-9	7	-3
3	13	-10	11	1

17. Determine the point on the graph of $y = 2\sqrt{x}$ that is closest to point (6, 0).

18. Let $h(x) = [g(f(x))]^2$, where f and g are continuous functions. Use the table on the left to evaluate $h(-1)$ and $h'(-1)$.

19. **Check Your Understanding:** Determine the slope of the tangent line to the graph of $f(x) = \left(\frac{x}{x+1}\right)^4$ at point (0, 0).

20. Let $h(x) = f(g(x))$. Prove that $h''(x) = f''(g(x))[g'(x)]^2 + g''(x)f'(g(x))$.

C **21.** Let $f(x) = \frac{1}{1+x}$ and $g(x) = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1+x}}}}$.

(a) Express $g(x)$ as a composition.

(b) Use the decomposition above to determine $g'(x)$.

22. **Thinking, Inquiry, Problem Solving:** The illumination of a point is inversely proportional to the square of the distance from the point to the light source. Two identical 10-m high light posts are 30 m apart. A person walks from one post to the other at 1 m/s. Determine the point where the illumination is greatest. At what point between the posts is the illumination weakest?

23. Find a so that the curve $y = \sqrt{ax^2 - 4}$ has a tangent with slope 2 at the point where $x = 2$.

ADDITIONAL ACHIEVEMENT CHART QUESTIONS

Knowledge and Understanding: Given $y = \frac{5x}{\sqrt{(2x+3)^3}}$, find $\frac{dy}{dx}$.

Application: A rectangle has dimensions $(2x + 3)^4$ and $(5 - \sqrt{6x})^3$. The rectangle's area increases or decreases, depending on the value of x. Find the rate of change of the area of the rectangle when $x = 5$. Determine whether the area is increasing or decreasing as x increases.

Thinking, Inquiry, Problem Solving: Suppose that k is a rational number, where $h(x) = x^{-k}$ and $f(x) = (x + k)^k$. Show $\frac{d}{dx}[h(x) \cdot f(x)] = \frac{d}{dx}[h(x)] \cdot \frac{d}{dx}[f(x)]$. Find two other pairs of functions whose derivatives have the same property.

Communication: A student found the following problem in his mother's old calculus textbook: "Find a formula for $\frac{d}{dx}[h(g(r(x)))]$." How would you express this formula in words or in a diagram so you can quickly remember how to find the derivative?

5. $(3u^2 + 6)(8x^3 + 6x)$

6. (a) $192(x - 2)^2$ (b) $(40x + 10)(4x^2 + 2x - 3)^4$

(c) $-\frac{6}{(12 - 6x)^{\frac{4}{5}}}$ (d) $\frac{3x^2 - 8x + 6}{2\sqrt{x^3 - 4x^2 + 6x}}$

(e) $\frac{2x - 2}{3(x^2 - 2x)^{\frac{2}{3}}}$ (f) $3x\sqrt{x^2 - 1}$

(g) $\frac{-4(4x + 7)}{(2x^2 + 7x - 6)^5}$ (h) $\frac{2}{(5 - x)^3}$

(i) $\frac{60(2 + x^2)}{(6x + x^3)^6}$ (j) $\frac{21x^2 - 2x}{2(7x^3 - x^2)^{\frac{1}{2}}}$

(k) $\frac{-(24x^3 - 2)}{(6x^4 - 2x)^{\frac{6}{5}}}$ (l) $\frac{3(2 + \sqrt{x})^2}{2\sqrt{x}}$

7. -6

8. $f'(0)$ undefined

9. -4; 35

10. $-\frac{2}{x^2}\left(\frac{1}{x} - 3\right)$

11. -32

12. $x < -1$

13. $\frac{1}{30}$

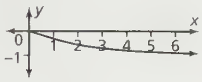
14. (a) $v(t) = \frac{2}{3}(t^2 + t)^{-\frac{1}{3}}(2t + 1)$; $a(t) = \frac{2}{9}(t^2 + t)^{-\frac{4}{3}}(2t^2 + 2t - 1)$

(b) 1.931 m/s (c) 2.360 m/s

(d) undefined (e) 0.141 m/s^2

15. (a) -0.5 m/s (b) 0 m/s

(c)



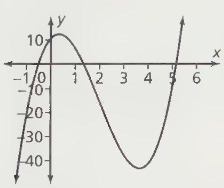
16. $f'(x) = \frac{2}{(\sqrt{x^2 + 1})^3}, f''(x) = \frac{6x}{(\sqrt{x^2 + 1})^5}$

17. As time goes by, the object travels towards, but never reaches, a point $\frac{5}{2}$ units to the right of the origin. Initially, its velocity is 5, but gradually decreases toward zero while remaining positive over time. Its acceleration also gradually decreases toward zero.

18. $(2, 0), (-2, 0), \left(\frac{2}{\sqrt{5}}, \frac{4096}{125}\right), \left(-\frac{2}{\sqrt{5}}, \frac{4096}{125}\right)$

19. $2x(x^2 - 1)(3x^2 + 7)$

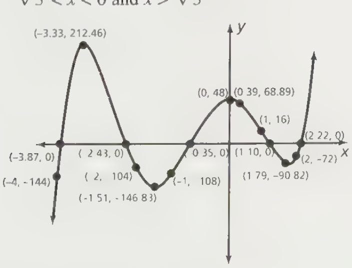
20.



21. (a) $x \in \mathbf{R}$ (b) $\left\{x \mid x \neq \frac{3}{2}, x \in \mathbf{R}\right\}$

22. $-\sqrt{3} < x < 0$ and $x > \sqrt{3}$

23.



24. b^{10}

25. $a = 2.3^{\frac{5}{3}}, b = 3^{\frac{2}{3}}, c = 9$

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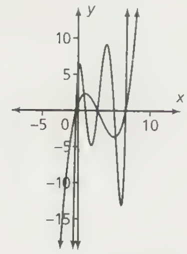
1. (a) $2s$ (b) $-\frac{2}{u^2}$ (c) -1 (d) $\frac{8}{9}$

2.



3. (a) $g(x) = f(h(x))$, where $h(x) = x^2 - 3x$ (b) $-\frac{2x - 3}{(x^2 - 3x)^2}$

4. (a) $f(x) = u + 5u^{-1}$, where $u = 2x - 3$; $g(x) = \sqrt{u} + 5u$, where $u = 2x - 3$

(b) $2[1 - 5(2x - 3)^{-2}]$

5. (a) $g(x) = h(f(x))$, where $f(x) = 2x - 3$, $h(x) = \sqrt{x} + 5x$

(b) $\frac{1}{\sqrt{2x - 3}} + 10$

6. (a) $6(2x - 5)^2(3x^2 + 4)^4(13x^2 - 25x + 4)$

(b) $\frac{8x^2(4x^2 + 2x - 3)^4}{52x^2 + 16x - 9}$

(c) $2(5 + x)(4 - 7x^3)^5(-70x^3 - 315x^2 + 4)$

(d) $\frac{6(-9x + 7)}{(3x + 5)^5}$ (e) $\frac{2(2x^2 - 5)^2(4x^2 + 48x + 5)}{(x + 8)^3}$

(f) $\frac{-3x^3(7x - 16)}{4(x - 2)^{\frac{3}{2}}}$ (g) $\frac{8(2x + 5)^3(x + 3)(x + 2)}{(6 - x^2)^5}$

(h) $-9(4x + x^2)^{-10}(4 + 2x)$

(i) $2(3x + 2)^{-\frac{1}{2}}(2x - x^3)^{3}[-33x^3 - 24x^2 + 18x + 16]$

(j) $\frac{-2x}{(x^2 + 1)^{\frac{1}{2}}(x^2 - 1)^{\frac{3}{2}}}$

(k) $5[2x + (3x^2 - 5x)^3]^4[2 + 3(3x^2 - 5x)^2(6x - 5)]$

(l) $\frac{2 + \frac{3}{2}\sqrt{x}}{2\sqrt{2x + x^{\frac{3}{2}}}}$

7. (a) $\frac{1}{2\sqrt{x}(\sqrt{x} + 1)^2}$ (b) $\frac{5}{2(2x + 1)^{\frac{1}{2}}(x + 3)^{\frac{3}{2}}}$

8. $3\left\{1 - \frac{6x + 11}{[x - 3(x + 2)^2]^2}\right\}$

9. $-2 - 5[t^{-2} - (3t + 5)^4]^4[-2t^{-3} - 12(3t + 5)^3]$

11. $-\frac{4}{3}$

12. $-2\{1 - [2x - 5(x - 2)^3]^{-2}[2 - 15(x - 2)^2]\}$

13. $\frac{3(3x - 1)}{\sqrt{(3x - 1)^2 - 1}}$

14. $y = 1472x - 2816$

15. $x < -1$ and $x > 1$

16. $(2, 0), \left(\frac{13}{2}, \frac{1}{6\sqrt{2}}\right)$

17. (4, 4)

18. 16; -280

19. 0

21. (a) $g(x) = f \circ f \circ f \circ f(x)$

(b) $g'(x) = (f'(f(f(f(x)))))(f'(f(f(x))))(f'(f(x)))(f'(x)) =$
$[1 + (1 + (1 + (1 + x)^{-1})^{-1})^{-1}]^{-2}$
$[1 + (1 + (1 + x)^{-1})^{-1}]^{-2}[1 + (1 + x)^{-1}]^{-2}[1 + x]^{-2}$

22. greatest: 0.309 m and 29.691 m; weakest: 15 m

23. $a = 2$