6.4 Finding Optimal Values for Composite Function Models

SETTING THE STAGE

You have used calculus to solve many real-world problems involving polynomial, rational, and now composite functions. In these problems, you are often asked to minimize distance, time, or cost. For example,

George wants to run a power line to a new cottage being built on an island that is 400 m from the shore of a lake. The main power line ends 3 km away from the point on the shore that is closest to the island. The cost of laying the power line under water is twice the cost of laying the power line on land. How should George place the line to minimize the overall cost?

In this section, you will revisit the techniques you learned for solving optimization problems in Chapters 4 and 5. You will apply these techniques to composite function models.

EXAMINING THE CONCEPT

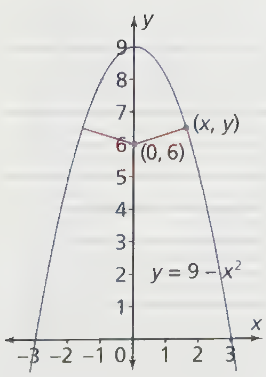
Solving Optimization Problems Involving the Chain Rule

Before solving the power line problem, consider some simpler examples. Recall the strategy for solving optimization problems in section 4.5 on page 305.

Example 1 Minimizing a Distance

Which points on the graph of $y = 9 - x^2$ are closest to point (0, 6)?

Solution



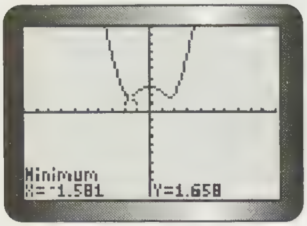
Graph the function and point (0, 6). The sketch shows that there are two points at a minimum distance from (0, 6).

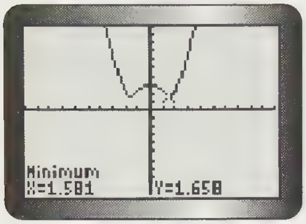
Any point on the graph can be represented by $(x, 9 - x^2)$. The distance, d, between this point and (0, 6) is given by $d(x) = \sqrt{(x - 0)^2 + [(9 - x^2) - 6]^2}$.

After simplifying, $d(x) = \sqrt{x^4 - 5x^2 + 9}$. You can find the point on the graph that is closest to (0, 6) by finding the value of x that minimizes

$d(x) = \sqrt{x^4 - 5x^2 + 9}$ or $d(x) = (x^4 - 5x^2 + 9)^{\frac{1}{2}}$.

You could use graphing technology to estimate the minimum value. Use calculus to solve the problem algebraically and determine the exact solution.





$$d'(x) = \frac{d}{dx}\left[(x^4 - 5x^2 + 9)^{\frac{1}{2}}\right]$$ Apply the chain rule.

$$= \frac{d(x^4 - 5x^2 + 9)^{\frac{1}{2}}}{d(x^4 - 5x^2 + 9)} \cdot \frac{d}{dx}(x^4 - 5x^2 + 9)$$

$$= \frac{1}{2}(x^4 - 5x^2 + 9)^{-\frac{1}{2}} \cdot \frac{d}{dx}(x^4 - 5x^2 + 9)$$

$$= \frac{1}{2(x^4 - 5x^2 + 9)^{\frac{1}{2}}} \cdot (4x^3 - 10x)$$ Factor the numerator.

$$= \frac{2x(2x^2 - 5)}{2(x^4 - 5x^2 + 9)^{\frac{1}{2}}}$$ Simplify.

$$= \frac{x(2x^2 - 5)}{\sqrt{x^4 - 5x^2 + 9}}$$

The derivative is a rational expression. To find the critical numbers, determine where $d'(x) = 0$ or $d'(x)$ does not exist. The only case in which $d'(x)$ cannot exist is when the denominator, $\sqrt{x^4 - 5x^2 + 9}$, is equal to 0. However, this case cannot occur, as is shown below.

$$x^4 - 5x^2 + 9 = \left(x^2 - \frac{5}{2}\right)^2 + \frac{11}{4}$$ Complete the square.

For all $x \in \mathbf{R}$, $\left(x^2 - \frac{5}{2}\right)^2 \geq 0$.

$$\therefore \left(x^2 - \frac{5}{2}\right)^2 + \frac{11}{4} > 0$$

$$\therefore \sqrt{x^4 - 5x^2 + 9} > 0.$$

Thus, $d'(x)$ is defined for all values of $x \in \mathbf{R}$. Also, $d'(x) = 0$ when the numerator, $x(2x^2 - 5)$, equals 0; that is, when $x = 0$ or $x = \pm\sqrt{\frac{5}{2}}$.

Solving $x = 0$ and $2x^2 - 5 = 0$ produces the critical numbers. Therefore, the critical numbers are $x = -\sqrt{\frac{5}{2}}$, $x = 0$, and $x = \sqrt{\frac{5}{2}}$. Apply the first derivative test to find the critical numbers that represent a local minimum.

	Intervals			
	$x < -\sqrt{\frac{5}{2}}$	$-\sqrt{\frac{5}{2}} < x < 0$	$0 < x < \sqrt{\frac{5}{2}}$	$x > \sqrt{\frac{5}{2}}$
x	−	−	+	+
$(2x^2 - 5)$	+	−	−	+
$\sqrt{x^4 - 5x^2 + 9}$	+	+	+	+
$d'(x)$	$\frac{(-)(+)}{(+)} = -$	$\frac{(-)(-)}{(+)} = +$	$\frac{(+)(-)}{(+)} = -$	$\frac{(+)(+)}{(+)} = +$
$d(x)$	decreasing ↘	increasing ↗	decreasing ↘	increasing ↗
	minimum at $x = -\sqrt{\frac{5}{2}}$	maximum at $x = 0$		minimum at $x = \sqrt{\frac{5}{2}}$

The first derivative test verifies that a local minimum occurs when $x = -\sqrt{\frac{5}{2}}$ or $x = \sqrt{\frac{5}{2}}$. These values correspond to the minimum distance. Substitute them into the original function $y = 9 - x^2$ to get the minimum distance, which is the corresponding value of the function.

$$y = 9 - \left(-\sqrt{\frac{5}{2}}\right)^2 \quad \text{and} \quad y = 9 - \left(\sqrt{\frac{5}{2}}\right)^2$$
$$= 9 - \frac{5}{2} \qquad\qquad = 9 - \frac{5}{2}$$
$$= \frac{13}{2} \qquad\qquad = \frac{13}{2}$$

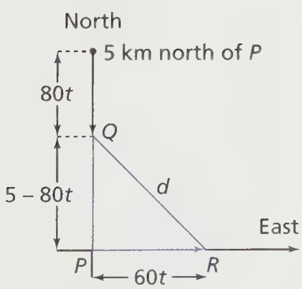
The two points closest to (0, 6) on the graph of $y = 9 - x^2$ are $\left(-\sqrt{\frac{5}{2}}, \frac{13}{2}\right)$ and $\left(\sqrt{\frac{5}{2}}, \frac{13}{2}\right)$.

Note that in this case, the calculator cannot be used to determine an exact solution, but can be used to check the reasonableness of the solution by evaluating $\sqrt{\frac{5}{2}}$ and comparing with the graphs above.

Example 2 Deciding When Two Moving Objects Are Closest to Each Other

A north–south highway intersects an east–west highway at point P. A vehicle crosses P at 1:00 p.m., travelling east at a constant speed of 60 km/h. At the same instant, another vehicle is 5 km north of P, travelling south at 80 km/h. Find the time when the two vehicles are closest to each other and the distance between them at that time.

Solution



Draw a diagram. Let the x- and y-axes be the highways, and let P be at the origin. The slower vehicle is travelling away from P, while the faster vehicle is travelling toward P.

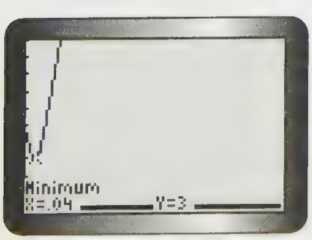
Let t be the number of hours after 1:00 P.M.

At time t, the slower vehicle is $60t$ kilometres east of P, and the faster vehicle is $(5 - 80t)$ kilometres north of P.

The quantity to be minimized is the distance between the two vehicles, d. From the diagram and using the Pythagorean theorem,

$$d^2 = (5 - 80t)^2 + (60t)^2$$
$$d^2 = 25 - 800t + 10\,000t^2$$
$$d = \sqrt{25 - 800t + 10\,000t^2}$$

$$\therefore d(t) = (25 - 800t + 10\,000t^2)^{\frac{1}{2}}$$



You can estimate the solution graphically, as shown. The minimum distance, approximately 3 km, occurs at $t = 0.04$.

Use calculus to calculate the exact solution algebraically. Applying the chain rule and the power rule,

$$d'(t) = \frac{d(25 - 800t + 10\,000t^2)^{\frac{1}{2}}}{d(25 - 800t + 10\,000t^2)} \cdot \frac{d}{dt}(25 - 800t + 10\,000t^2)$$

$$= \frac{1}{2}(25 - 800t + 10\,000t^2)^{-\frac{1}{2}} \cdot \frac{d}{dt}(25 - 800t + 10\,000t^2)$$

$$= \frac{1}{2}(25 - 800t + 10\,000t^2)^{-\frac{1}{2}} \cdot (-800 + 20\,000t)$$

$$= \frac{-800 + 20\,000t}{2(25 - 800t + 10\,000t^2)^{\frac{1}{2}}}$$

$$= \frac{-400 + 10\,000t}{\sqrt{25 - 800t + 10\,000t^2}}$$

Critical numbers occur when $d'(t) = 0$ or is undefined. Note that the denominator equals $\sqrt{(100t - 4)^2 + 9}$, which is always greater than 0; therefore, the derivative is always defined. So, $d'(t) = 0$ when the numerator equals 0.

$$\therefore 0 = -400 + 10\,000t$$
$$t = 0.04$$

Use the first derivative test to verify that $t = 0.04$ represents a local minimum.

	Intervals	
	$0 \le t < 0.04$	$t > 0.04$
$-400 + 10\,000t$	$-$	$+$
$\sqrt{25 - 800t + 10\,000t^2}$	$+$	$+$
$d'(t)$	$\frac{(-)}{(+)} = -$	$\frac{(+)}{(+)} = +$
$d(t)$	decreasing ↘	increasing ↗
	minimum at $x = 0.04$	

The first derivative test confirms that a local minimum occurs when $t = 0.04$. In this case, the calculator gives the exact solution.

The distance between the vehicles is minimized at this time. The value $t = 0.04$ means 0.04 h, or $0.04 \times 60 = 2.4$ min, or 2 min 24 sec.

Evaluate d when $t = 0.04$ to determine the minimum distance.

$$d(0.04) = \sqrt{25 - 800(0.04) + 10\,000(0.04)^2}$$
$$= 3$$

The two vehicles are closest to each other when they are 3 km apart at exactly 1:02:24 P.M.

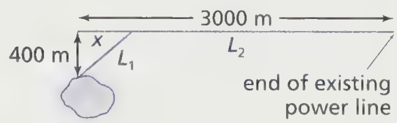
Example 3 Minimizing the Cost of an Electrical Power Line

Recall the problem in Setting the Stage.

George wants to run a power line to a new cottage being built on an island that is 400 m from the shore of a lake. The main power line ends 3 km away from the point on the shore that is closest to the island. The cost of laying the power line under water is twice the cost of laying the power line on land. How should George place the line to minimize the overall cost?

Solution

Sketch a diagram of the situation. The cable will run along the shore, enter the water at some point, and continue straight to the island.

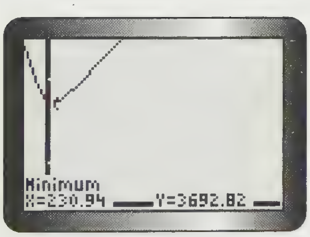


Let x represent the distance between the point on the shore closest to the island and the point where the cable enters the water. Let L_1 represent the length of cable underwater and L_2 the length of cable on land. Therefore, the total cost to lay the cable is $C = 2L_1 + L_2$. Minimize this quantity, C.

From the diagram, $L_2 = 3000 - x$ and $(L_1)^2 = x^2 + 400^2$, so $L_1 = \sqrt{x^2 + 160\ 000}$. Substituting in the expression for C gives

$$C(x) = 2(\sqrt{x^2 + 160\ 000}) + (3000 - x)$$

$$\therefore\ C(x) = 2(x^2 + 160\ 000)^{\frac{1}{2}} + (3000 - x) \qquad 0 \le x \le 3000$$



You can estimate the minimum value using graphing technology.

Now use calculus to calculate the exact solution algebraically.
Using the difference and chain rules,

$$C'(x) = \frac{d}{dx}\left[2(x^2 + 160\ 000)^{\frac{1}{2}}\right] + \frac{d}{dx}(3000) - \frac{d}{dx}(x)$$

$$C'(x) = \frac{d[2(x^2 + 160\ 000)^{\frac{1}{2}}]}{d(x^2 + 160\ 000)} \cdot \frac{d(x^2 + 160\ 000)}{dx} + 0 - 1$$

$$= \left[2\left(\frac{1}{2}\right)(x^2 + 160\ 000)^{-\frac{1}{2}} \cdot \frac{d}{dx}(x^2 + 160\ 000)\right] - 1$$

$$= \left[(x^2 + 160\ 000)^{-\frac{1}{2}} \cdot 2x\right] - 1$$

$$= \frac{2x}{\sqrt{x^2 + 160\ 000}} - 1$$

Critical numbers occur when $C'(x)$ equals 0 or is undefined. In this case, the denominator is always greater than 0, so $C'(x)$ is defined for all x where $0 < x < 3000$. Solve for x to determine where $C'(x) = 0$.

$$\frac{2x}{\sqrt{x^2 + 160\ 000}} - 1 = 0$$ Add 1 to each side.

$$\frac{2x}{\sqrt{x^2 + 160\ 000}} = 1$$ Multiply each side by $\sqrt{x^2 + 160\ 000}$.

$$2x = \sqrt{x^2 + 160\ 000}$$ Square both sides.

$$4x^2 = x^2 + 160\ 000$$ Isolate x^2.

$$3x^2 = 160\ 000$$ Solve for x.

$$x = \pm\sqrt{\frac{160\ 000}{3}}$$

$$x \doteq 230.94$$ Since $x \geq 0$, the negative solution is inadmissible.

	Intervals	
	$0 \leq x < 230.94$	$230.94 < x \leq 3000$
$C'(x)$	−	+
$C(x)$	decreasing ↘	increasing ↗
	minimum at $x = 230.94$	

The first derivative test confirms that a global minimum occurs at $x \doteq 230.94$. This value is the distance from the point on the shore closest to the island to the point where the cable enters the water. This distance will minimize the cost of installing the cable. Using this value, calculate L_1 and L_2.

$$L_1 \doteq \sqrt{(230.94)^2 + 160\ 000} \doteq 461.88 \qquad L_2 \doteq 3000 - 230.94 = 2769.06$$

The total cost of laying the cable is minimized if about 2769.06 m are placed along the shore and 461.88 m are placed underwater.

CHECK, CONSOLIDATE, COMMUNICATE

1. Why do optimization problems involving distance often result in using the chain rule?
2. Suppose you have found the model for a problem. Is determining the optimal values of a composite function different from determining the optimal values for polynomial and rational functions? Explain your answer in terms of techniques.

KEY IDEAS

- The properties of the derivative that you use to solve optimization problems involving composite functions are the same as those you apply to polynomial and rational functions.
- Use the strategy for solving optimization problems, outlined in Chapters 4 and 5 on pages 305 and 397, to also solve problems involving composite function models.

6.4 Exercises

A

1. Find the point on the line $y = 6x + 8$ that is closest to the origin, (0, 0).

2. What point on the graph of $f(x) = x^2$ is closest to (2, 0.5)?

3. Find the point on the curve $2y^2 = 4(x + 1)$ that is closest to the origin.

4. Ship A, sailing due east at 8 km/h, sights ship B 5 km to the southeast when ship B is sailing due north at 6 km/h. How close to each other will the two ships be when they pass?

5. Mike, who is standing on the deck of a yacht that is travelling due west at 6 km/h, sees a sailboat sailing southwest at 4 km/h, 3 km northwest of the yacht. How close to each other do these boats get?

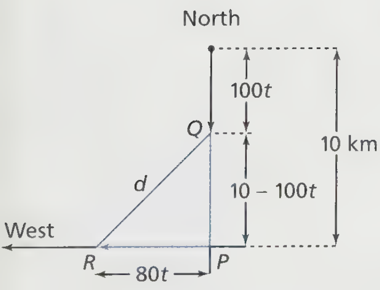
6. **Knowledge and Understanding:** Determine the points on the curve $y = \sqrt{x}$, $0 \le x \le 1$, that are **(a)** closest to and **(b)** farthest from point (2, 0).

B

7. A passenger jet is travelling due north at 400 km/h, while a cargo plane, 2 km to the northwest, is travelling east at 300 km/h. The altitude of the cargo plane is 1000 m lower than the altitude of the passenger jet. What will be the minimum separation between the two aircraft?

8. A truck travelling west at 100 km/h is 250 km due east of a sports car going north at 120 km/h. When will the vehicles be closest to each other? What is the minimum distance between them?

9. A boat leaves a dock at 2:00 P.M., heading west at 15 km/h. Another boat heads south at 12 km/h and reaches the same dock at 3:00 P.M. When were the boats closest to each other?



10. **Communication:** Devise an optimization problem based on the diagram on the left.

11. A woman lives on an island 2 km from the mainland. Her fitness club is 4 km along the shore from the point closest to the island. To get to the fitness club, she paddles her kayak at 2 km/h. Once she reaches the shore, she jogs at 4 km/h. Determine where she should land to reach her fitness club in the shortest possible time.

12. A dune buggy is on a straight desert road, 40 km north of Dustin City. The vehicle can travel at 45 km/h off the road and 75 km/h on the road. The driver wants to get to Gulch City, 50 km east of Dustin by another straight road, in the shortest possible time. Determine the route he should take.

13. Suppose you are swimming in a lake and find yourself 200 m from shore. You would like to get back to the spot where you left your towel, which is at least 200 m down the beach, as quickly as possible. You can walk along the beach at 100 m/min, but you can swim at only 50 m/min. To what point on the beach should you swim to minimize your total travelling time?

14. The owners of a small island want to bring in electricity from the mainland. The island is 80 m from a straight shoreline at the closest point. The nearest electrical connection is 200 m along the shore from that point. It costs twice as much to install cable across water than across land. What is the least expensive way to install the cable?

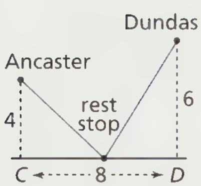
15. **Application:** Bill owns an oil well located 400 m from a road. Bill wants to connect the well to a storage tank 1200 m down the road from the well. It costs \$35/m to lay pipe along the road and \$50/m to lay it elsewhere. How should the pipeline be laid to minimize the total cost?

16. **Thinking, Inquiry, Problem Solving:** Find the dimensions of the rectangle of maximum area that can be inscribed in the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

17. **Check Your Understanding:** Is the strategy for solving optimization problems different for rational and composite function models? Explain.

C **18.** An offshore oil well is in the ocean at a point 5 km from the closest point on a straight shoreline. The oil must be pumped from the well to a storage facility on shore. The storage facility is 15 km away from the point on shore that is closest to the well. It costs \$100 000/km to lay pipe underwater and \$75 000/km over land. How should the pipeline be situated on the shore to minimize the cost?

19. Prove that (1, 0) is the closest point on the circle $x^2 + y^2 = 1$ to (2, 0).



20. Two towns, Ancaster and Dundas, are 4 km and 6 km, respectively, from an old railroad line that has been made into a bike trail. Points C and D on the trail are the closest points to the two towns, respectively. These points are 8 km apart. Where should a rest stop be built to minimize the length of new trail that must be built from both towns to the rest stop?

ADDITIONAL ACHIEVEMENT CHART QUESTIONS

Knowledge and Understanding: A north–south highway intersects an east–west highway at point P. A vehicle passes P at 3:00 P.M., travelling east at a constant speed of 100 km/h. At the same instant, another vehicle passes 2 km north of P, travelling south at 80 km/h. Find the time when the two vehicles are closest to each other and when the distance between them is minimal.

Application: Millie is in a boat 2 km from a straight shoreline and wants to reach a point on the shore 10 km north of her present position, as shown in the diagram. She can row at 5 km/h and jog at 8 km/h. Calculate the position on the shoreline to which she should row to reach the destination in the shortest time.

Thinking, Inquiry, Problem Solving: Find the area of the largest rectangle that can be inscribed in a circle with radius r.

Communication: To solve an optimization problem, you set the derivative of the function that models the problem to 0 and solve the resulting equation. Why?

6.4 Exercises, page 471

1. $\left(-\frac{48}{37}, \frac{8}{37}\right)$
2. $\left[\left(\frac{4}{3}\right)^{\frac{1}{3}}, \left(\frac{4}{3}\right)^{\frac{2}{3}}\right]$
3. $(-1, 0)$
4. 0.707 km
5. 0.171 km
6. (a) (1, 1) (b) (0, 0)
7. 1.7381 km
8. 1.02 h; 192.06 km
9. 2:23.25
10. Example: A north-south highway intersects an east-west highway at a point P. A vehicle crosses P at 12 a.m., travelling west at a constant speed of 100 km/h. At the same instant, another vehicle is 10 km north of P, travelling south at 80 km/h. Find the time when the two vehicles are closest to each other and the distance between them at that time.
11. 1.15 km $\left(\text{or } \frac{2}{\sqrt{3}} \text{ km}\right)$ down the coast
12. He should drive off road directly to the point on the road 20 km west of Gulch City, then drive east the rest of the way.
13. 115.5 km down the coast
14. lay 92.4 m of cable below water and 153.8 m along the shoreline
15. lay 807.9 m of pipe along the road and 560 m of pipe elsewhere
16. $4\sqrt{2} \times 3\sqrt{2}$
17. no; take the derivative of an equation representing what needs to be found with respect to the variable one needs to optimize and solve for that variable by setting the derivative equal to 0
18. 5.67 km down the shoreline
20. 5.12 km from Ancaster; 7.68 km from Dundas

6.5 Exercises, page 478

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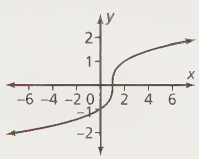

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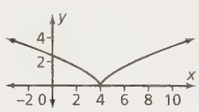

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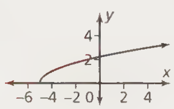

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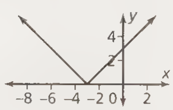

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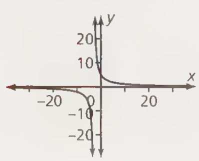

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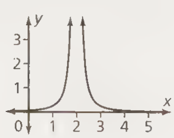

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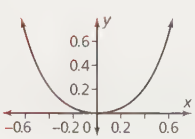

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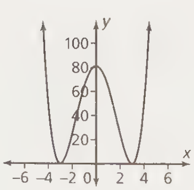

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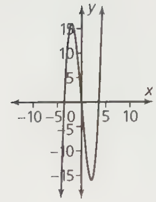

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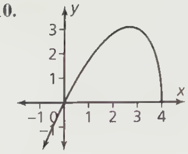

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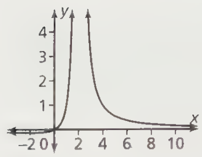

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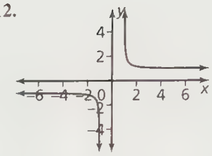

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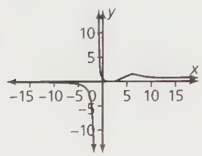

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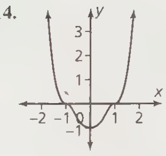
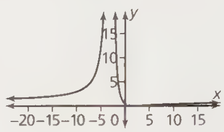


15.


6.6 Exercises, page 483

1. (a) $3x^2$ (b) $20x^3$ (c) $\frac{dy}{dx}$ (d) $3y^2\frac{dy}{dx}$ (e) $30y^5\frac{dy}{dx}$
2. (a) $y + x\frac{dy}{dx}$ (b) $8x^3y + 2x^4\frac{dy}{dx}$
   (c) $2xy^2 + 2x^2y\frac{dy}{dx}$ (d) $10xy^3 + 15x^2y^2\frac{dy}{dx}$
   (e) $-8x^3y^6 - 12x^4y^5\frac{dy}{dx}$
3. $-\frac{7}{2}$
5. (a) $\frac{-x}{y}$ (b) $\frac{x}{y}$ (c) $-\frac{x^2}{y^2}$ (d) $-\sqrt{\frac{y}{x}}$
   (e) $-\left[\frac{2x-y}{2y-x}\right]$ (f) $-\left[\frac{y^2+2xy}{x^2+2xy}\right]$
   (g) $\frac{1-2xy^3}{1+3x^2y^2}$ (h) $\frac{8xy-3(x^2+y^2)}{6xy-4x^2}$
6. (a) 4 (b) $-\frac{1}{2}$ (c) $-\frac{8}{3}$
   (d) $-\frac{5}{7}$ (e) $-\frac{27}{11}$ (f) 0
7. $y = -4x + 1.5$
8. $\frac{3}{4}$
9. $-\frac{1}{8}$
11. 10:01:12 a.m.
13. -3.30 units/s
14. $\frac{dy}{dx} = \pm\frac{x}{\sqrt{16-x^2}}$ or $\frac{dy}{dx} = \frac{-x}{y}$; preferred method will vary
15. $\frac{27}{20}$
16. $4\sqrt{x+y} - 1$
17. $\frac{4}{(x+y)^2} - 1$
18. (a) $0, \frac{1}{3}$ (b)