6.5 Sketching Graphs of Composite Functions

SETTING THE STAGE

In previous chapters, you used calculus to analyze and sketch the graphs of polynomial and rational functions. The first derivative helps identify the intervals in the domain where the function increases or decreases and locate any maximum or minimum values. Use the second derivative to determine the concavity of the graph, along with points of inflection. Recall that rational functions have asymptotes, while polynomial functions do not. Composite functions may or may not have asymptotes.

In this section, you will use the first and second derivatives to analyze the behaviour of composite functions.

EXAMINING THE CONCEPT

Analyzing and Graphing a Composite Function

Recall these steps for graphing a polynomial function or a rational function, $f(x)$.

1. From the equation for $f(x)$,
- determine the domain and any discontinuities
- determine the intercepts
- find any asymptotes

2. From the equation for $f'(x)$,
- find the critical numbers
- determine the intervals of increase and decrease
- identify any local maximum or minimum values

3. From the equation for $f''(x)$,
- determine where the graph is concave up and where it is concave down
- find any points of inflection

4. Use the information from steps 1 to 3 to graph the function.

You will apply these same steps to graph composite functions.

Example 1 Analyzing the Graph of a Radical Function

(a) For $f(x) = \sqrt{x^2 - 4}$, determine $f'(x)$ and $f''(x)$.

(b) Discuss the domain, asymptotes, critical numbers, intervals of increase or decrease, local extrema, concavity, and points of inflection.

(c) Graph the function.

Solution

(a) $f(x) = \sqrt{x^2 - 4}$ Express the function with a rational exponent.

$= (x^2 - 4)^{\frac{1}{2}}$ Differentiate using the chain rule.

$$f'(x) = \frac{d(x^2 - 4)^{\frac{1}{2}}}{d(x^2 - 4)} \cdot \frac{d}{dx}(x^2 - 4)$$

$$= \frac{1}{2}(x^2 - 4)^{-\frac{1}{2}} \cdot \frac{d}{dx}(x^2 - 4)$$

$= \frac{1}{2}(x^2 - 4)^{-\frac{1}{2}}(2x)$ Simplify.

$= \frac{x}{\sqrt{x^2 - 4}}$ Express as a product so you can use the product rule.

$$= (x^2 - 4)^{-\frac{1}{2}}(x)$$

$f''(x) = \frac{d}{dx}\left[(x^2 - 4)^{-\frac{1}{2}}\right] \cdot (x) + \frac{d}{dx}(x) \cdot (x^2 - 4)^{-\frac{1}{2}}$ Apply the product rule.

$= -\frac{1}{2}(x^2 - 4)^{-\frac{3}{2}}(2x)(x) + (1)(x^2 - 4)^{-\frac{1}{2}}$ Apply the chain rule with the power rule.

$= -\frac{1}{2}(x^2 - 4)^{-\frac{3}{2}}[2x^2 - 2(x^2 - 4)]$ Factor out $-\frac{1}{2}(x^2 - 4)^{-\frac{3}{2}}$.

$$= \frac{-4}{\sqrt{(x^2 - 4)^3}}$$

1. Analyze $f(x)$.

(b) Since $f(x) = \sqrt{x^2 - 4}$, f is undefined when $x^2 - 4 < 0$.

The domain of f is $\{x \mid |x| \geq 2, x \in \mathbf{R}\}$. The x-intercepts are 2 and -2.

As $x \to \pm\infty$, $f(x) \to \infty$. There are no asymptotes.

2. Analyze $f'(x)$.

Critical numbers occur when $f'(x) = 0$ or when $f'(x)$ is undefined. In this case, $f'(x) = \frac{x}{\sqrt{x^2 - 4}}$, suggesting that the critical numbers are ± 2 and 0.

Recall that $f(x)$ is undefined for $-2 < x < 2$, so 0 does not have to be considered.

Therefore, the intervals of increase and decrease are $x < -2$ and $x > 2$. Apply the first derivative test to locate any local extrema.

	Intervals	
	$x < -2$	$x > 2$
x	$-$	$+$
$\sqrt{x^2 - 4}$	$+$	$+$
$f'(x)$	$\frac{(-)}{(+)} = -$	$\frac{(+)}{(+)} = +$
$f(x)$	decreasing ↘	increasing ↗

For $x < -2$, $f(x)$ is decreasing. For $x > 2$, $f(x)$ is increasing. Since $f(x)$ is undefined for $-2 < x < 2$, and $f(x)$ decreases to $x = -2$ and increases beyond $x = 2$, $f(-2) = 0$ and $f(2) = 0$ are absolute minima.

3. Analyze $f''(x)$.

Now consider the second derivative and concavity. Possible points of inflection occur when $f''(x) = \frac{-4}{\sqrt{(x^2-4)^3}} = 0$ or when $f''(x)$ is undefined, that is, at $x = \pm 2$.

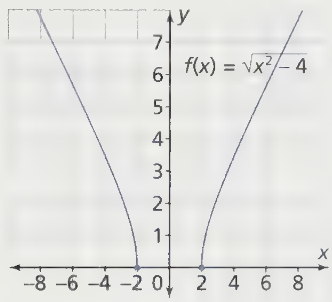
Analyze the second derivative on the same intervals to determine where the graph of $f(x)$ is concave up or concave down.

	Intervals	
	$x < -2$	$x > 2$
-4	$-$	$-$
$\sqrt{(x^2-4)^3}$	$+$	$+$
$f''(x)$	$\frac{(-)}{(+)} = -$	$\frac{(-)}{(+)} = -$
$f(x)$	concave down	concave down

When $x < -2$ and $x > 2$, the graph of $f(x)$ is concave down. There are no points of inflection.

4. Sketch the graph.

(c)



Example 2 Analyzing the Graph of a Rational Function

(a) For $f(x) = \frac{18(1-x)}{(x+3)^2}$, determine $f'(x)$ and $f''(x)$.

(b) Determine the domain, asymptotes, critical numbers, intervals of increase or decrease, local extrema, concavity, and points of inflection.

(c) Graph the function.

Solution

(a)

$$f(x) = \frac{18(1-x)}{(x+3)^2} = 18(1-x)(x+3)^{-2}$$

Express the fraction as a product so you can use the product rule.

Determine the first derivative.

$$\begin{aligned} f'(x) &= 18\left(\frac{d}{dx}(1-x)\cdot(x+3)^{-2} + \frac{d}{dx}[(x+3)^{-2}]\cdot(1-x)\right) \\ &= 18[(-1)(x+3)^{-2} + (-2)(x+3)^{-3}(1-x)] \\ &= 18[-(x+3)^{-2} - 2(x+3)^{-3}(1-x)] \\ &= 18(x+3)^{-3}[-(x+3) - 2(1-x)] \\ &= 18(x+3)^{-3}(x-5) \end{aligned}$$

Determine the second derivative.

$$f''(x) = 18\left(\frac{d}{dx}[(x + 3)^{-3}] \cdot (x - 5) + \frac{d}{dx}(x - 5) \cdot (x + 3)^{-3}\right)$$
$$= 18[-3(x + 3)^{-4}(x - 5) + (1)(x + 3)^{-3}]$$
$$= 18(x + 3)^{-4}[-3(x - 5) + (x + 3)]$$
$$= 18(x + 3)^{-4}(-2x + 18)$$
$$= -36(x + 3)^{-4}(x - 9)$$

1. Analyze $f(x)$.

(b) The domain is $\{x \mid x \neq -3, x \in \mathbf{R}\}$. $f(1) = 0$, so the graph crosses the x-axis at $x = 1$.

As $x \to \pm\infty$, $f(x) \to 0$. Therefore, the x-axis is the horizontal asymptote.

If $x > 1$, then $f(x) < 0$. Therefore, as $x \to \infty$, the graph must approach the x-axis from below.

As $x \to -\infty$, $f(x) > 0$. Therefore, the graph must approach the x-axis from above as $x \to -\infty$. $f(-3)$ is undefined.

As $x \to -3$ (from either side), $f(x) > 0$ and $f(x) \to \infty$. Therefore, there is a vertical asymptote at $x = -3$.

2. Analyze $f'(x)$.

The critical numbers occur when $f(x)$ is defined and $f'(x) = \frac{18(x - 5)}{(x + 3)^3} = 0$ or when $f'(x)$ is undefined. In this case, the critical number may be $x = 5$ since a vertical asymptote occurs at $x = -3$. (Note that, as $x \to \pm\infty$, $f'(x) \to 0$, which confirms that there is a horizontal asymptote.)

The vertical asymptote and the critical number divides the domain into three intervals, $x < -3$, $-3 < x < 5$, and $x > 5$.

Apply the first derivative test for the intervals in the table to find any local extrema.

	Intervals		
	$x < -3$	$-3 < x < 5$	$x > 5$
$18(x - 5)$	−	−	+
$(x + 3)^3$	−	+	+
$f'(x)$	$\frac{(-)}{(-)} = +$	$\frac{(-)}{(+)} = -$	$\frac{(+)}{(+)} = +$
$f(x)$	increasing ↗	decreasing ↘	increasing ↗
	vertical asymptote at $x = -3$		minimum at $x = 4$

When $x < -3$ and $x > 5$, $f(x)$ is increasing. When $-3 < x < 5$, $f(x)$ is decreasing.

A local maximum does not exist at $x = -3$ since the line $x = -3$ is a vertical asymptote. By the first derivative test, there is a local minimum at $x = 5$, and $f(5) = \frac{18(1 - 5)}{(5 + 3)^2} = -1.125$.

3. Analyze $f''(x)$.

Now consider the second derivative. Possible points of inflection occur when $f''(x) = -\frac{36(x-9)}{(x+3)^4} = 0$ or when $f''(x)$ is undefined, that is, at $x = -3$ and $x = 9$. However, $f(-3)$ is undefined, so there can be no point of inflection there.

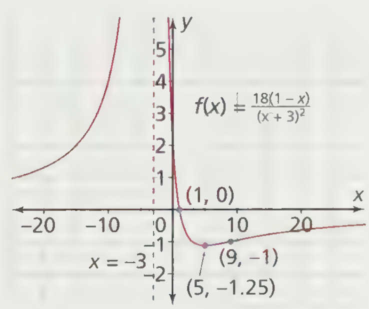
Analyze the second derivative to determine concavity.

	Intervals		
	$x < -3$	$-3 < x < 9$	$x > 9$
$-36(x-9)$	$+$	$+$	$-$
$(x+3)^4$	$+$	$+$	$+$
$f''(x)$	$\frac{(+)}{(+)} = +$	$\frac{(+)}{(+)} = +$	$\frac{(-)}{(+)} = -$
$f(x)$	concave up	concave up	concave down
	vertical asymptote at $x = -3$		point of inflection at $x = 9$

When $x < 9$, $x \neq -3$, the graph of $f(x)$ will be concave up. When $x > 9$, the graph of $f(x)$ will be concave down. The graph will have a point of inflection at $x = 9$. Since $f(9) = \frac{18(1-9)}{(9+3)^2} = -1$, the point of inflection is $(9, -1)$.

4. Sketch the graph.

(c)



CHECK, CONSOLIDATE, COMMUNICATE

1. Do all composite functions have asymptotes? Explain.
2. Can you analyze and sketch the graph of a composite function in the same way that you analyze and sketch the graph of a polynomial function or a rational function?
3. Which differentiation rule will you always need to determine the first and second derivatives of a composite function?

KEY IDEAS

- The method used for sketching the curves of polynomial and rational functions is the same as for sketching the curve of a composite function.
- When determining the first and second derivatives of any composite function, use the chain rule.
- The graphs of composite functions may or may not have asymptotes. Examine the function carefully to be sure that you have checked for asymptotes in your analysis.

6.5 Exercises

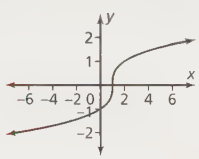
A For each function, use the techniques shown in this section to sketch the graph of the function. Use $f(x)$ to find the domain, intercepts, and asymptotes. Use $f'(x)$ to find the critical numbers, intervals of increase or decrease, and local extrema. Use $f''(x)$ to find concavity and points of inflection.

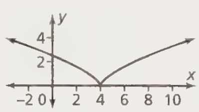
1. $f(x) = (x - 1)^{\frac{1}{3}}$
2. $f(x) = (x - 4)^{\frac{2}{3}}$
3. $f(x) = \sqrt{x - 5}$
4. $f(x) = \sqrt{(x + 3)^2}$
5. $f(x) = \dfrac{15}{x + 3}$
6. $f(x) = (2x - 4)^{-2}$
7. $f(x) = (x^3 + x)^2$
8. $f(x) = (x^2 - 9)^2$
9. $f(x) = x(x^2 - 12)$
10. $f(x) = x\sqrt{4 - x}$
11. $f(x) = \dfrac{x}{(x - 2)^2}$
12. $f(x) = \dfrac{x}{\sqrt{x^2 - 1}}$
13. $f(x) = \dfrac{(x - 1)^2}{(x + 1)^3}$
14. $f(x) = (x^2 + 1)^2(x^2 - 1)^3$
15. $f(x) = \left(\dfrac{x - 2}{x + 3}\right)^2$

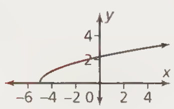
6.4 Exercises, page 471

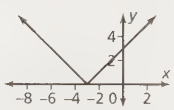
1. $\left(-\frac{48}{37}, \frac{8}{37}\right)$
2. $\left[\left(\frac{4}{3}\right)^{\frac{1}{3}}, \left(\frac{4}{3}\right)^{\frac{2}{3}}\right]$
3. $(-1, 0)$
4. 0.707 km
5. 0.171 km
6. (a) (1, 1) (b) (0, 0)
7. 1.7381 km
8. 1.02 h; 192.06 km
9. 2:23.25
10. Example: A north-south highway intersects an east-west highway at a point P. A vehicle crosses P at 12 a.m., travelling west at a constant speed of 100 km/h. At the same instant, another vehicle is 10 km north of P, travelling south at 80 km/h. Find the time when the two vehicles are closest to each other and the distance between them at that time.
11. 1.15 km $\left(\text{or } \frac{2}{\sqrt{3}} \text{ km}\right)$ down the coast
12. He should drive off road directly to the point on the road 20 km west of Gulch City, then drive east the rest of the way.
13. 115.5 km down the coast
14. lay 92.4 m of cable below water and 153.8 m along the shoreline
15. lay 807.9 m of pipe along the road and 560 m of pipe elsewhere
16. $4\sqrt{2} \times 3\sqrt{2}$
17. no; take the derivative of an equation representing what needs to be found with respect to the variable one needs to optimize and solve for that variable by setting the derivative equal to 0
18. 5.67 km down the shoreline
20. 5.12 km from Ancaster; 7.68 km from Dundas

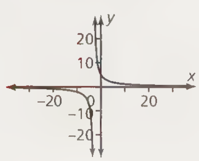
6.5 Exercises, page 478

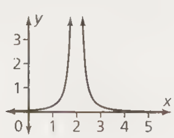
1. 

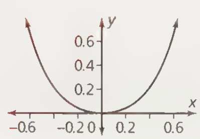
2. 

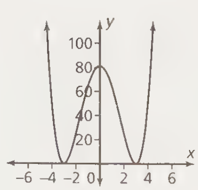
3. 

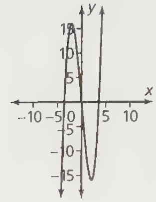
4. 

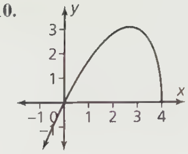
5. 

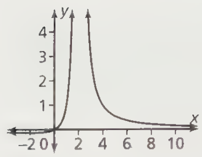
6. 

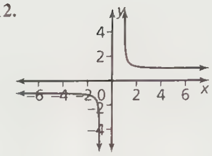
7. 

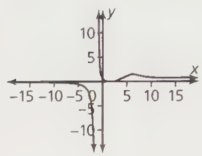
8. 

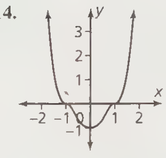
9. 

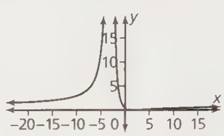
10. 

11. 

12. 

13. 

14. 

15. 

6.6 Exercises, page 483

1. (a) $3x^2$ (b) $20x^3$ (c) $\frac{dy}{dx}$ (d) $3y^2\frac{dy}{dx}$ (e) $30y^5\frac{dy}{dx}$
2. (a) $y + x\frac{dy}{dx}$ (b) $8x^3y + 2x^4\frac{dy}{dx}$
   (c) $2xy^2 + 2x^2y\frac{dy}{dx}$ (d) $10xy^3 + 15x^2y^2\frac{dy}{dx}$
   (e) $-8x^3y^6 - 12x^4y^5\frac{dy}{dx}$
3. $-\frac{7}{2}$
5. (a) $\frac{-x}{y}$ (b) $\frac{x}{y}$ (c) $-\frac{x^2}{y^2}$ (d) $-\sqrt{\frac{y}{x}}$
   (e) $-\left[\frac{2x-y}{2y-x}\right]$ (f) $-\left[\frac{y^2+2xy}{x^2+2xy}\right]$
   (g) $\frac{1-2xy^3}{1+3x^2y^2}$ (h) $\frac{8xy-3(x^2+y^2)}{6xy-4x^2}$
6. (a) 4 (b) $-\frac{1}{2}$ (c) $-\frac{8}{3}$
   (d) $-\frac{5}{7}$ (e) $-\frac{27}{11}$ (f) 0
7. $y = -4x + 1.5$
8. $\frac{3}{4}$
9. $-\frac{1}{8}$
11. 10:01:12 a.m.
13. -3.30 units/s
14. $\frac{dy}{dx} = \pm\frac{x}{\sqrt{16-x^2}}$ or $\frac{dy}{dx} = \frac{-x}{y}$; preferred method will vary
15. $\frac{27}{20}$
16. $4\sqrt{x+y} - 1$
17. $\frac{4}{(x+y)^2} - 1$
18. (a) $0, \frac{1}{3}$ (b)