6.7 Related Rate Models

SETTING THE STAGE

Explore the concepts in this lesson in more detail in Exploration 13 on page 584.

You have used the chain rule to find $\frac{dy}{dx}$ implicitly. You can also use the chain rule to find the rates of change of two or more related quantities that are changing with respect to time. For example, as water drains from a funnel, the volume of water, V, the radius of the water's surface, r, and the level of the water, h, are all functions of time. These three quantities are also related to one another.

Joanne conducts an experiment by letting water drain from a cone-shaped funnel. She uses a graduated cylinder and a timer to measure how quickly the water drains from the funnel. Joanne wants to know the rate at which the water level in the funnel dropped.

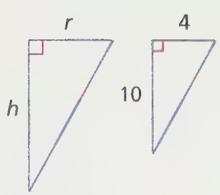
In this section, you will use the chain rule to solve problems involving related rates of change.

EXAMINING THE CONCEPT

Solving Related Rate Problems

Joanne's experiment involves **related rates**. The rate at which the water level drops is related to the rate at which the water drains from the funnel, which has a radius of 5.44 cm and a height of 13.6 cm. In related rate problems, you often have to find the rate of change of one variable given the rate of change of some other variable. Understanding the relations between the variables allows you to solve these problems.

Thirty seconds after Joanne's experiment starts, the water level in the funnel is 10 cm, the radius of the water's surface is 4 cm, and the water is draining at a rate of 2.5 mL/s. How fast is the water level dropping?



$\frac{r}{4} = \frac{h}{10}$

$\therefore r = \frac{4}{10}h$

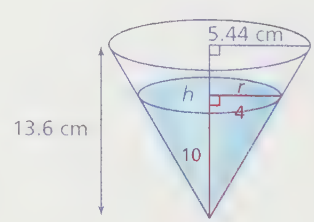
$r = \frac{2}{5}h$

Let V represent the volume of water in the funnel, h the depth of water, r the radius of the surface, and t the time in seconds.

Given: At $t = 30$ s, $\frac{dV}{dt} = -2.5$ mL/s. (The rate is negative since volume is decreasing with time.)

Evaluate: $\frac{dh}{dt}$ at $t = 30$.

The variables are related by the formula for the volume of a cone, $V = \frac{\pi}{3}r^2h$. At $t = 30$ s, $h = 10$ cm and $r = 4$ cm. By the properties of cones and similar triangles, $r = \frac{2}{5}h$, for any positive value of h. Therefore, you can simplify the volume formula so that it has only one variable.



$$V = \frac{\pi}{3}r^2h$$
$$V = \frac{\pi}{3}\left(\frac{2}{5}h\right)^2 h$$
$$V = \frac{\pi}{3}\left(\frac{4}{25}h^2\right)h$$
$$V = \frac{4\pi}{75}h^3$$
$$\doteq 0.168h^3$$

Use this formula to find a relation between $\frac{dV}{dt}$ and $\frac{dh}{dt}$.

$$V = 0.168h^3$$ Differentiate both sides with respect to t.

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$$ Apply the chain rule.

$$\frac{dV}{dt} = \frac{d}{dh}(0.168h^3) \cdot \frac{dh}{dt}$$
$$\frac{dV}{dt} = (0.504h^2)\frac{dh}{dt}$$

Now substitute the known values for $\frac{dV}{dt}$ and h at $t = 30$ s.

$$-2.5 = 0.504(10^2)\left(\frac{dh}{dt}\right)$$ Solve for $\frac{dh}{dt}$.

$$\frac{dh}{dt} \doteq -0.05$$

Thus, at 30 s, the water level in the funnel is dropping at a rate of about 0.05 cm/s.

Strategy for Solving Related Rate Problems

- Define variables for any quantities that vary with time. Define distances relative to a fixed point, where possible. Identify other quantities as constants.
- Draw and label a diagram showing all variables and constants.
- Express all given information in terms of variables and the derivatives of variables. Express the rate to be determined as a derivative.
- Find an equation that relates the variables used in the given derivatives and the derivative to be determined. This may require finding several equations and then substituting.
- Substitute all *constant* quantities into the equation *before* differentiating.
- Differentiate with respect to time, and then evaluate the derivative for a specific instant in time.

A common mistake in solving related rate problems is to substitute variable information before differentiating. Substitute only constant values before differentiating.

Example 1 **Ripples in a Pond**

A pebble is dropped into a pond. The resulting ripples form concentric circles. The radius of the outermost circle increases at the constant rate of 10 cm/s. Determine the rate at which the area of disturbed water is changing when the radius is 50 cm.

Solution

Let r represent the radius of the circle of disturbed water and A the area of this circle.

Given: $\frac{dr}{dt} = 10$ cm/s (The rate is positive since the radius is increasing with time.)

Evaluate: $\frac{dA}{dt}$ when $r = 50$ cm.

In this case, variables r and A are related by the formula for the area of a circle, $A = \pi r^2$.

$A = \pi r^2$	Differentiate both sides with respect to t.
$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$	Apply the chain rule.
$\frac{dA}{dt} = \frac{d}{dr}(\pi r^2) \cdot \frac{dr}{dt}$	
$= 2\pi r \cdot \frac{dr}{dt}$	Substitute $\frac{dr}{dt} = 10$.
$= 2\pi r(10)$	
$= 20\pi r$	This is the derivative at any time.

If $r = 50$, then $\frac{dA}{dt} = 20\pi(50) = 1000\pi$.

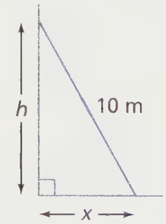
When the radius is 50 cm, the area of the disturbed water is increasing at a rate of 1000π cm^2/s.

Example 2 Related Rates Involving Distance

Sandy is standing on a 10-m ladder leaning against a wall when the foot of the ladder starts to slide away from the wall at 0.5 m/s. Determine the rate at which the top of the ladder is sliding down the wall when the foot of the ladder is 6 m from the wall.

Solution

Sketch a diagram. Define variables for distances, relative to a fixed point.



Let h represent the height of the top of the ladder and x the distance from the wall to the foot of the ladder. Both x and h are measured in metres. Let t represent the time in seconds since the ladder started to slip.

Given: $\frac{dx}{dt} = 0.5$ m/s (The rate is positive because x is increasing.)

Evaluate: $\frac{dh}{dt}$ when $x = 6$ m.

Find an equation relating x and h. You will want to differentiate this equation with respect to t to show a relation between $\frac{dh}{dt}$ and $\frac{dx}{dt}$.

Using the Pythagorean theorem,

$$10^2 = h^2 + x^2$$ Differentiate both sides with respect to t.

$$\frac{d}{dt}(10^2) = \frac{d}{dt}(h^2) + \frac{d}{dt}(x^2)$$ Use the chain rule.

$$0 = \frac{d(h^2)}{dh} \cdot \frac{dh}{dt} + \frac{d(x^2)}{dx} \cdot \frac{dx}{dt}$$

$$0 = 2h\frac{dh}{dt} + 2x\frac{dx}{dt}$$ Substitute the known value, $\frac{dx}{dt} = 0.5$.

$$0 = 2h\frac{dh}{dt} + 2x(0.5)$$ Isolate $\frac{dh}{dt}$.

$$\frac{dh}{dt} = -\frac{x}{2h}$$

To evaluate the derivative when $x = 6$, as required, solve for the missing variable h. Use $10^2 = h^2 + x^2$.

$$10^2 = h^2 + 6^2$$
$$h^2 = 64$$
$$h = 8 \text{ (since } h \geq 0)$$

If $x = 6$, then

$$\frac{dh}{dt} = -\frac{6}{2(8)} = -\frac{3}{8}$$

The negative value for the derivative means that the height is decreasing as time increases. When the foot of the ladder is 6 m from the wall, the top of the ladder is falling at $\frac{3}{8}$ m/s.

Example 3 Another Related Rate Problem Involving Distance

A police cruiser is approaching an intersection when the officer hears a report of a car speeding along the cross street. The police car is 100 m from the intersection, travelling at 30 m/s. The other car is 200 m past the intersection, travelling at 27 m/s. The roads are at right angles. At what rate is the distance between the two cars closing?

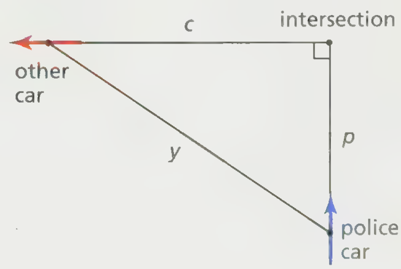
Solution

Let t represent the time in seconds since the car crossed the intersection.

Let p represent the distance in metres from the police car to the intersection.

Let c represent the distance from the intersection to the other car.

Let y represent the distance between the two vehicles.



Given: $\frac{dc}{dt} = 27$ m/s, $\frac{dp}{dt} = -30$ m/s ($\frac{dc}{dt}$ is positive since c is increasing, and $\frac{dp}{dt}$ is negative since p is decreasing.)

Evaluate: $\frac{dy}{dt}$ when $c = 200$ m and $p = 100$ m.

Find an equation relating c, p, and y. Use the Pythagorean theorem.

$$y^2 = c^2 + p^2.$$ Differentiate both sides with respect to t.

$$\frac{d}{dt}(y^2) = \frac{d}{dt}(c^2) + \frac{d}{dt}(p^2)$$ Apply the chain rule.

$$\frac{d}{dy}(y^2) \cdot \frac{dy}{dt} = \frac{d(c^2)}{dc} \cdot \frac{dc}{dt} + \frac{d(p^2)}{dp} \cdot \frac{dp}{dt}$$

$$2y\frac{dy}{dt} = 2c\frac{dc}{dt} + 2p\frac{dp}{dt}$$ Divide both sides by 2. Substitute the known constants, $\frac{dc}{dt} = 27$ and $\frac{dp}{dt} = -30$.

$$y\frac{dy}{dt} = c(27) + p(-30)$$ Isolate $\frac{dy}{dt}$.

$$\frac{dy}{dt} = \frac{27c - 30p}{y}$$

Use the values of c, p, and y described in the problem to evaluate the derivative. The values of c and p are $c = 200$ m and $p = 100$ m. Calculate the value of y.

$$y^2 = 200^2 + 100^2$$
$$y = \sqrt{50\ 000}$$
$$\doteq 223.6$$ (since $y \geq 0$)

Now substitute into the derivative $\frac{dy}{dt}$.

$$\frac{dy}{dt} = \frac{27(200) - 30(100)}{223.6}$$
$$\doteq 10.7$$

The positive value means the distance between the two vehicles is increasing, not decreasing. At the moment when the police car is 100 m from the intersection, the distance between the vehicles is increasing by about 10.7 m/s.

Example 4

Related Rates and Volume

At 1:00 P.M. on a hot summer day, the temperature is 27°C and rising by 2°C/h. A balloon filled with air has a volume of 0.01 m^3. The air pressure inside the balloon remains constant at 120 kPa because the balloon expands or contracts, depending on the temperature. Determine the rate at which the volume of the balloon is increasing at 1:00 P.M. Note that $\frac{PV}{T} = k$, where P is the pressure, V is the volume, T is temperature in kelvin (27°C = 300K), and k is a constant.

Solution

Let t represent the elapsed time in hours after 1:00 P.M.

Given: $P = 120$ kPa, and when $t = 0$, $\frac{dT}{dt} = 2$, $V = 0.01\ m^3$, and $T = 300$ K

Evaluate: $\frac{dV}{dt}$ when $t = 0$

Starting with $\frac{PV}{T} = k$, at $t = 0$, we have $P = 120$, $T = 300$, and $V = 0.01$.

$$\therefore \frac{120(0.01)}{300} = k$$
$$0.004 = k$$ where k is constant

Note that all the constant values were substituted to find a relation between the variables in the problem.

We were given $\frac{dT}{dt}$ at a specific time and asked to determine $\frac{dV}{dt}$ at that time.

There is a relation between V and t, so differentiate with respect to t.

The other constant value is $P = 120$.

$$\therefore \frac{120V}{T} = 0.004$$

Thus, at any time t, we have $V = \frac{0.004}{120}T = \frac{T}{30\,000}$.

From $V = \frac{T}{30\,000}$, Differentiate both sides with respect to t. The result gives the derivative of V with respect to t in terms of another derivative.

$$\frac{dV}{dt} = \frac{dV}{dT} \cdot \frac{dT}{dt}$$

$$\frac{dV}{dt} = \frac{1}{30\,000} \cdot \frac{dT}{dt}$$

At $t = 0$, it is given that $\frac{dT}{dt} = 2$.

Therefore,

At $t = 0$, $$\frac{dV}{dt} = \frac{1}{30\,000}(2) = \frac{1}{15\,000}$$

You cannot assume that $\frac{dT}{dt}$ is constant. You only know its value at exactly 1:00 P.M., in this case, $t = 0$. Similarly, the value for $\frac{dV}{dt}$ is valid only at $t = 0$.

At 1:00 P.M., the volume of the balloon is increasing at $\frac{1}{15\,000}$ m^3/h.

CHECK, CONSOLIDATE, COMMUNICATE

1. Why is $xy = 6$ and $\frac{dx}{dt} = 3$ enough information to determine $\frac{dy}{dt}$ when $x = 6$?
2. If $\frac{dy}{dt} = -2$, why is it incorrect to conclude that y is always decreasing at a rate of -2?

KEY IDEAS

- To solve related rate problems, do the following:
  - identify and define all relevant variables (quantities that vary with time)
  - define distances relative to a fixed point, where possible
  - draw a diagram
  - express all given and required information in terms of the variables and derivatives of the variables
  - find an equation (or equations) relating the variables that appear in the given or required derivatives
  - substitute constant values and relations between variables
  - differentiate with respect to time
  - substitute information that is valid at a specific time to evaluate the derivative
  - carefully interpret the signs of derivatives

6.7 Exercises

A For questions 1 to 4, assume that x and y are differentiable functions of t. Then determine the indicated values of $\frac{dy}{dt}$ and $\frac{dx}{dt}$.

Equation:	Find:	Given:
1. $y = \sqrt{x}$	**(a)** $\frac{dy}{dt}$ when $x = 3$	$\frac{dx}{dt} = 4$
	(b) $\frac{dx}{dt}$ when $x = 30$	$\frac{dy}{dt} = 2$
2. $y = x^2 - 3x$	**(a)** $\frac{dy}{dt}$ when $x = 2$	$\frac{dx}{dt} = 3$
	(b) $\frac{dx}{dt}$ when $x = 1$	$\frac{dy}{dt} = 5$
3. $xy = 4$	**(a)** $\frac{dy}{dt}$ when $x = 10$	$\frac{dx}{dt} = 20$
	(b) $\frac{dx}{dt}$ when $x = 5$	$\frac{dy}{dt} = -2$
4. $x^2 + y^2 = 9$	**(a)** $\frac{dy}{dt}$ when $x = 3, y = 0$	$\frac{dx}{dt} = 6$
	(b) $\frac{dx}{dt}$ when $x = 0, y = -3$	$\frac{dy}{dt} = -2$

5. Let $y\frac{dx}{dt} + 2x\frac{dy}{dt} = 6$. What information do you need to evaluate $\left.\frac{dy}{dt}\right|_{x=2}$?

6. Determine $\frac{dy}{dt}$ if $x + 2y = 4$ and $\frac{dx}{dt} = 3$.

B

7. **(a)** Determine an expression for $\frac{dy}{dt}$ if $x^2 + 2y^2 = 8$ and $\frac{dx}{dt} = 3$.

(b) What additional information would you need to evaluate $\frac{dy}{dt}$?

8. Point P moves from left to right along the x-axis at 2 units/second. At the instant when $x = 4$, at what rate is the point directly above P on the parabola $y = x^2 - 3x + 5$ moving up or down?

9. **Knowledge and Understanding:** A point is moving along the right branch of a hyperbola defined by $4x^2 - y^2 = 64$. What is $\frac{dy}{dt}$ when the point is at $(5, -6)$ and $\frac{dx}{dt} = 3$?

10. Differentiate each relation with respect to time.

(a) $A = \pi r^2$ **(b)** $V = \frac{4}{3}\pi r^3$ **(c)** $A = 4\pi r^2$

(d) $x^2 + y^2 = r^2$ **(e)** $V = \pi r^2 h$ **(f)** $A = 2\pi r^2 + 2\pi rh$

11. How fast is the area of a circular oil spill increasing when its radius is 50 m, if the radius is increasing at 0.1 m/s?

12. The radius of a melting snowball is decreasing at a rate of 2 cm/h. Determine the rate at which the volume of snow is decreasing when the radius is 10 cm.

13. The surface area of a sphere is decreasing at a constant rate of 2π cm^2/s. At what rate is the volume of the sphere decreasing at the instant when the radius is 2 cm?

14. A conical icicle melts at a rate of 1.2 cm^3/h. At 10:00 A.M., the icicle is 25 cm long and 4 cm in diameter at its widest point. The icicle keeps the same proportions as it melts. Determine the rate at which its length is decreasing at 10:00 A.M.

15. **Communication:** When an inflated balloon is released, it expels air at a nearly constant rate. Describe how the balloon accelerates. Support your answer as completely as possible.

16. **Application:** A circular oil spill is spreading so that $\frac{dr}{dt} = 1430r^{-1.5}$, where r is the radius in metres of the area covered in oil at t hours since the spill occurred. Determine the rate at which the area is growing when the area is 500.0 m^2.

17. Sand forms a conical pile as it falls from a conveyor belt onto the ground. The diameter of the base of the cone is always four times the height of the pile. The sand is falling at 0.1 m^3/s.

    (a) Graph the volume of the sand in the pile versus time. Justify your graph.

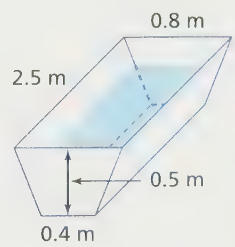
    (b) Graph the height of the pile versus time. Justify your graph.

    (c) Determine $\frac{dh}{dt}$.

    (d) Explain how your formula in (c) is consistent with your graph in (b).

    (e) How fast is the height increasing when the pile contains 200 m^3 of sand?

18. A trough has an isosceles trapezoidal cross section as shown in the diagram. Water is draining from the trough at 0.2 m^3/s.



    (a) At what rate is the surface area of the water decreasing?

    (b) At what rate is the water's depth decreasing when the depth is 0.2 m?

19. An image of a box on a computer screen is changing shape so that the length is decreasing at 2 cm/s and the width is increasing at 8 cm/s. Originally, the box was 20 cm long, 12 cm wide, and 16 cm high. The front face retains its original proportions as the box changes shape.

(a) Determine a formula for the rate of change of the area of the front face.

(b) Determine a formula for the rate of change of the volume of the box.

(c) Determine the rate at which the volume is changing after 5 s.

(d) State the formula for the rate of change of volume with respect to time.

(e) Sketch a graph of volume versus time.

20. A line is defined by $x + 2y = 10$. A second line is defined by $x - 2y = 5k$, where k increases by 2 units/second. P is the point where the two lines intersect.

(a) Describe the motion of P.

(b) Determine the rate at which the x-coordinate of P is increasing when P is at $Q(8, 1)$.

(c) Determine the rate at which the y-coordinate of P is increasing when P is at Q.

(d) Determine the coordinates of P exactly 0.01 s after it passes Q.

(e) Determine the distance P moves in the 0.01 s after it passes Q.

(f) Estimate the rate at which P is moving as it passes Q.

(g) Can you think of another way to determine the rate at which P is moving when it passes Q?

21. The table lists the annual unit sales and the number of employees at a Canadian cell phone company over five years.

Year	Cell Phones Sold	Employees
1997	56 000	121
1998	62 000	115
1999	58 000	110
2000	53 000	102
2001	40 000	85

(a) Develop an algebraic model for cell phone sales as a function of the year. Explain.

(b) You might expect the number of employees to be a function of the number of cell phones sold. Determine a reasonable algebraic model for this interpretation.

(c) Express the number of employees as a function of the year, using your answers to the previous parts of this question. Compare this model with the actual values.

(d) Express the instantaneous rate of change in the number of employees as a function of time. Compare this model with the actual values.

(e) Are these models reasonable? Justify your answer.

22. The relation between the number of products sold, n, and the price, $\$p$, is modelled by $pn = k$, where k is a constant. When Marissa set the price of her lemonade to 50¢ a glass, she sold 100 glasses.

(a) How many glasses would she sell if she raised the price to 75¢ a glass?

(b) The cost of the ingredients to make a glass of lemonade is a function of time, modelled by $c(t) = -10^{-6}t(t - 365)(t - 50) + 15$, where c is the cost per glass in cents and t is the day of the year. Is this model reasonable? Explain.

23. A strong wind is blowing when Eduardo is flying a kite. He notices that the kite string will maintain a constant angle with the ground (about 60°) if he releases the string at 0.5 m/s. At what rate is the kite's altitude increasing when he has let out 100 m of string?

24. A weather balloon is rising vertically at a rate of 4 m/s. An observer stands 100 m from the point where the balloon was released. How fast is the distance between the observer and the balloon changing when the balloon is 120 m high?

25. At noon, a truck is 250 km due east of a car. The truck is travelling west at a constant speed of 60 km/h, while the car is travelling north at 100 km/h.

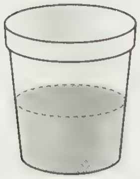
(a) At what rate is the distance between the vehicles changing at time t, in hours?

(b) What is the minimal distance between the car and the truck?

26. A sailor in a boat 10 km off a straight coastline wants to reach a point 20 km along the coast in the shortest possible time. Toward what point on the shore should he head if he can row at 4 km/h and run at 12 km/h?

27. A 1.7-m tall man walks away from a 9-m high streetlight at 0.8 m/s. How fast is the end of his shadow moving when he is 20 m from the lamppost?

28. **Thinking, Inquiry, Problem Solving:** A penny lies on the bottom of a 2-m deep pool. Jaclyn, whose eyes are 1.8 m above the surface of the water, sees the image of the penny 3.2 m from the edge of the pool, where she is standing. Fermat's principle states that light follows the path that minimizes the time to get from the object to the observer's eyes. The indices of light refraction in air and water are 1.02 and 1.333, respectively. The speed of light through a medium is its speed in a vacuum, c m/s, divided by the index of refraction. Determine the actual location of the penny on the bottom of the pool. Verify your answer using Snell's law.



29. **Check Your Understanding:** Outline a procedure for solving related rate problems.

C 30. Coffee is poured at a uniform rate of 3 cm^3/s into a cup whose inside is shaped like a truncated cone. The upper and lower radii of the cup are 4 cm and 2 cm, respectively. The height of the cup is 6 cm. How fast will the level of coffee be rising when the coffee is halfway up the cup? (Hint: Extend the diagram to form a cone.)

31. The altitude of a right circular cone increases at the same rate as the radius of its base decreases. At a certain instant, the altitude is 30 cm and the rate of change of the volume is 0. What is the base radius at this time?

ADDITIONAL ACHIEVEMENT CHART QUESTIONS

Knowledge and Understanding: The circumference of a circle is decreasing at a rate of 3 m/s. Determine the rate at which the area is changing when the circumference is 30 m.

Emilie De Breteuil (1706–1749)

Emilie de Breteuil translated Isaac Newton's work on calculus into French, which helped his ideas to gain importance in Europe. She was also a confidant of the French philosopher Voltaire.

Application: Two planes originated from the same airport. One plane is travelling at 260 km/h due east. The other is travelling at 230 km/h due south. What is the rate of change of the distance between the two planes when they are 30 km and 50 km, respectively, from the airport?

Thinking, Inquiry, Problem Solving: During a recent meteor storm, scientists observed that a meteor's burn rate was proportional to the surface area at any point in time as the meteor entered the atmosphere. Assuming that meteors are spherical in shape, how would you convince a classmate that the radius of each meteor was decreasing at a constant rate?

Communication: Hannah drew a diagram to represent two vehicles approaching an intersection at right angles. She differentiated the change in position of each vehicle over time and found the values to be negative at a particular time. She also differentiated the distance between the vehicles with respect to time and found this value to be negative at the same point in time. Explain what the negative values mean in both situations.

19. $\frac{3-x}{2y+6}$

20. **(a)** $\frac{dP}{dV} = -\frac{k}{V^2}$
(b) if volume is large, rate of change of pressure is closer to 0; but as volume increases, pressure decreases.

21. $-\frac{x+y}{x}$

23. $\sqrt{2a} \times \sqrt{2b}$

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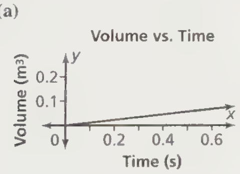
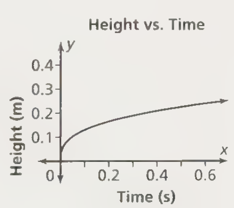
1. **(a)** $\frac{2}{\sqrt{3}}$ **(b)** $4\sqrt{30}$
2. **(a)** 3 **(b)** -5
3. **(a)** $-\frac{4}{5}$ **(b)** $\frac{25}{2}$
4. **(a)** does not exist **(b)** does not exist
5. y at $x = 2$ and $\frac{dx}{dt}$ at $x = 2$
6. $-\frac{3}{2}$
7. **(a)** $-\frac{3x}{2y}$ **(b)** a value of x and the value of y at that value of x
8. 10 units/s [up]
9. -10
10. **(a)** $\frac{dA}{dt} = 2\pi r\left(\frac{dr}{dt}\right)$ **(b)** $\frac{dV}{dt} = 4\pi r^2\left(\frac{dr}{dt}\right)$
(c) $\frac{dA}{dt} = 8\pi r\left(\frac{dr}{dt}\right)$ **(d)** $x\left(\frac{dx}{dt}\right) + y\left(\frac{dy}{dt}\right) = r\left(\frac{dr}{dt}\right)$
(e) $\frac{dV}{dt} = \pi\left[r^2\left(\frac{dh}{dt}\right) + 2rh\left(\frac{dr}{dt}\right)\right]$
(f) $\frac{dA}{dt} = 2\pi\left[(2r + h)\left(\frac{dr}{dt}\right) + r\left(\frac{dh}{dt}\right)\right]$
11. 10π m^2/s
12. 800π cm^3/s
13. 2π cm^3/s
14. 0.0955 cm/s
15. At first, balloon will accelerate very slowly; as it shrinks, air resistance will decrease and balloon will accelerate more quickly.
16. 2529.65 m^2/h
17. **(a)**

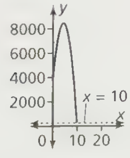


(b)



(c) $\frac{1}{40\pi h^2}$
(d) As height increases with respect to time, the slope of the graph of height vs. time decreases and approaches 0.
Hence, $\lim_{t \to \infty} \frac{dh}{dt} = \lim_{h \to \infty} \frac{1}{40\pi h^2} = 0$.
(e) 0.000 605 m/s

18. **(a)** $-\left(\frac{0.4}{2h+1}\right)$ m^2/s, where h = depth of water **(b)** $\doteq 0.018$ m/s
19. **(a)** $-3.2l$ **(b)** $3.2l(2l - w)$
(c) -1024 cm^3/s **(d)** $25.6(t - 10)(3t - 7)$
(e)



20. **(a)** P moves along the line $x + 2y = 10$
(b) 5 units/s **(c)** -2.5 units/s **(d)** (8.05, 0.975)
(e) 0.0559 units **(f)** 5.59 units/s
(g) determine the speed using the x-speed, y-speed, and the Pythagorean theorem
21. **(a)** $y = -2\,785.714x^2 + 11\,133\,185.71x - 1.112\,346 \times 10^{10}$
(b) $y = 0.001\,48x + 26.781$
(c) $-4.123x^2 + 16\,477.115x + 16\,462\,694.02$
(d) $-8.246x + 16\,477.115$
(e) Answers will vary.
22. **(a)** 67 glasses, to nearest glass
(b) yes, for all t; $c(t)$ rises to its max. in the summer, when lemonade is in higher demand, and it decreases to its min. in the winter, when lemonade is in less demand
23. 0.433 m/s
24. 3.073 m/s
25. **(a)** $\frac{-15\,000 + 13\,600t}{\sqrt{(-250 + 60t)^2 + (100t)^2}}$ km/h
(b) 214.4 km
26. 16.464 km along the shoreline from his destination
27. 0.986 m/s
28. 2.652 m from the edge of the pool
29. Identify all quantities and draw a labelled sketch; Write an equation with the variables whose rates of change are given or to be determined; Differentiate with respect to time using the chain rule; substitute and solve.
30. $\frac{1}{3\pi}$ cm/s
31. 60 cm

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13. Answers will vary. Example: $f(x) = x + 2$, $g(x) = x - 1$, $f \circ g(x) = x + 1$ with domain $\{1, 3, 4\}$
14. **(a)** x or $-x + 2$ **(b)** x
(c) f and g are inverses of each other
15. $D = \{x \neq -5, x \neq 2, x \in \mathbf{R}\}$
16. **(a)** $f'(x) = 5(4x^2 + 3x)^4(8x + 3)$
(b) $g'(x) = \frac{20x - 5}{2\sqrt{10x^2 - 5x}}$ **(c)** $h'(x) = \frac{21(12x^2 - 12x)}{(4x^3 - 6x^2)^4}$
(d) $y' = \frac{2}{3(x+5)^{\frac{1}{3}}}$
17. 2
18. increasing: $x < 0$; decreasing: $x > 0$
19. **(a)** $f'(x) = (6x^2 - 5x)^4(2x - 1)^3(168x^2 - 150x + 25)$
(b) $g'(x) = \frac{9x^3(7x - 2)}{\sqrt{3x^3 - x^2}}$
(c) $h'(x) = \frac{(2x + 5)(-56x^3 - 132x^2 + 180x)}{(4x^3 - 6x^2)^4}$
(d) $y' = \frac{5(3x^2 - 5)^4(3x^2 - 6x + 5)}{(x - 1)^6}$
20. $\frac{dP}{dt} = 0.5[2t + 5(t^2 - 5t)^3]^{-0.5}[2 + 15(t^2 - 5t)^2(2t - 5)]$
21. min.: $(0.655, -11.131)$; max.: $(-0.655, 11.131)$
22. (1, 1)
23. 9 min 41 s, 23.681 km
24. 5.24 m from the closest point of the conduit to the house (toward the direction of the main cable line)
25. $f'(x) = 3(x - 1)^2$,
$f''(x) = 6(x - 1)$